
Problem 1. (Finite maps.) Let \( f_0, \ldots, f_m \) be homogeneous polynomials of degree \( d > 0 \) without common zeros on \( X \subset \mathbb{P}^n \). Show that

\[
\begin{align*}
  f : X \to \mathbb{P}^m, & \quad f(x) = [f_0(x) : \ldots : f_m(x)]
\end{align*}
\]
gives a finite morphism onto its image.

Answer: Let \( v : \mathbb{P}^n \to \mathbb{P}^N \) be the Veronese embedding. Let \( F = v \circ f \). Since \( v \) is an isomorphism onto image,

\[
  f \text{ finite } \iff F \text{ finite}.
\]

Let \( \ell_0, \ldots, \ell_m \) be the linear polynomials corresponding to \( f_0, \ldots, f_m \) under Veronese. Then

\[
  F(x) = [\ell_0(x) : \ldots : \ell_m(x)].
\]

Without loss of generality, we may assume \( \ell_0, \ldots, \ell_m \) are independent. For otherwise, assuming \( \ell_0, \ldots, \ell_k \) are the independent ones and setting

\[
  \tilde{F}(x) = [\ell_0(x) : \ldots : \ell_k(x)]
\]

we have \( F = A \circ \tilde{F} \) for an injective linear map \( A : \mathbb{C}^k \to \mathbb{C}^\ell \). Since \( A \) is an isomorphism onto its image, it suffices to prove that \( \tilde{F} \) is finite. After a change of coordinates, we may assume \( \ell_0 = x_0, \ldots, \ell_k = x_k \). The map \( \tilde{F} \) becomes the composition of projections away from a point, which we have seen is finite. \( \square \)

Problem 2. (Semicontinuity of fiber dimensions.) Assume that \( f : X \to Y \) is a surjective morphism of projective varieties. Show that \( Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} \) is closed.

Answer: Let \( n = \dim X, m = \dim Y \). We will use induction on \( m \), the case \( m = 0 \) being clear. Let \( d = n - m \geq 0 \). When \( k \leq d \) there is nothing to prove since \( Y_k = Y \) by the theorem of dimension of fibers. Assume \( k > d \). Let \( U \) be an open subset over which \( \dim f^{-1}(y) = d \). The existence of \( U \) is proven in the theorem of dimension of fibers. Let \( Z = Y \setminus U \) is a closed subset of \( Y \), so \( \dim Z < \dim Y \implies \dim Z \leq m - 1 \). Note that \( Y_k \subset Z \) for \( k > d \). We wish to show that \( Y_k \) is closed in \( Z \), so then it will be closed in \( Y \) as well.

To this end, consider the restriction \( \tilde{f} : W \to Z \), where \( W = f^{-1}(Z) \). We clearly have

\[
  Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} = \{ y \in Z : \dim \tilde{f}^{-1}(y) \geq k \}.
\]

We use induction on the dimension of the base to conclude. Indeed, \( \dim Z \leq m - 1 \), and working with each irreducible component of \( Z \) one at a time, we may assume \( Z \) is irreducible. In this case however the domain \( W = f^{-1}(Z) \) may have components
$W_1, \ldots, W_r$. Since $Z$ is closed, $W_i$ are closed in $X$ hence projective in $X$. Let $\bar{f}_i : W_i \to Z$ be the restricted morphism. Then

$$Y_k = \bigcup Y_k(\bar{f}_i)$$

where the sets of the right are calculated with respect to the morphisms $\bar{f}_i$. By the induction hypothesis applied to $\bar{f}_i$, it follows that $Y_k(\bar{f}_i)$ are closed in the image of $\bar{f}_i$ which in turn is closed in $Z$ by projective hypothesis, hence $Y_k$ is closed in $Z$ as well. □

Problem 3. (Criterion for irreducibility.) Assume that $f : X \to Y$ is a surjective morphism of projective algebraic sets such that $Y$ is irreducible and all fibers of $f$ are irreducible of the same dimension. Show that $X$ is irreducible as well.

Answer: Write $n$ for the common dimension of the fibers. Write $X_1, \ldots, X_r$ for the irreducible components of $X$. Then

$$\bigcup f(X_i) = Y$$

and $f(X_i)$ are closed in $Y$ since $f$ is a morphism of projective sets, hence a closed map. Since $Y$ is irreducible, there exists $i$ such that $f(X_i) = Y$. Assume that $X_1, \ldots, X_s$ are chosen so that

$$f(X_1) = \ldots = f(X_s) = Y$$

but $f(X_j) \neq Y$ for $j > s$. Construct $U_1, \ldots, U_s$ nonempty open sets in $Y$ such that

$$y \in U_i, 1 \leq i \leq s \implies \dim(f|_{X_i})^{-1}(y) = n_i = \dim X_i - \dim Y.$$ 

In fact, even for $j > s$ we can define $U_j = Y \setminus f(X_j)$ and for $y \in U_j$ we have

$$(f|_{X_j})^{-1}(y) = \emptyset.$$ 

Write

$$U = \cap_{i=1}^r U_i,$$

which is open and nonempty. For $y \in U$, $f^{-1}(y)$ is irreducible and nonempty, and is covered by $X_1, \ldots, X_r$ so it must exist $i_0$ such that

$$f^{-1}(y) \subset X_{i_0}.$$ 

It is clear from the choice of $i_0$ that the entire fiber over $y$ can be computed in $X_{i_0}$ so that

$$(f|_{X_{i_0}})^{-1}(y) = f^{-1}(y) \neq \emptyset$$

so

$$f|_{X_{i_0}} : X_{i_0} \to Y$$

must be surjective by the definition of $U_{i_0}$, and $i_0 \leq s$. Furthermore $n = n_{i_0}$ is the common dimension of the fibers since the fiber dimension can be calculated at $y$ and

$$(f|_{X_{i_0}})^{-1}(y) = f^{-1}(y).$$ 

If $z \in Y$, then

$$(f|_{X_{i_0}})^{-1}(z) \subset f^{-1}(z)$$
and the left hand side is at least of dimension \( n_{i_0} = \dim X_{i_0} - \dim Y \) by the theorem on dimension of fibers. But \( f^{-1}(z) \) is irreducible and \( n = n_{i_0} \) dimensional, so must have equality. Thus

\[
f^{-1}(z) \subset X_{i_0}
\]

for all \( z \in Y \). This shows that there are no components in \( X \) other than \( X_{i_0} \), so \( X \) is irreducible.

\[\square\]

**Problem 4.** *(Intersections in projective space.)* Let \( X \) and \( Y \) be two subvarieties of \( \mathbb{P}^n \). Show that if \( \dim X + \dim Y \geq n \), then \( X \cap Y \) is not empty.

**Answer:** Let \( H_1, H_2 \) be two disjoint linear subspaces of dimension \( n \) in \( \mathbb{P}^{2n+1} \). We write \([x_0 : x_1 : \ldots : x_n : y_0 : y_1 : \ldots : y_n]\) for the homogeneous coordinates in \( \mathbb{P}^{2n+1} \). Without loss of generality, we may assume \( H_1 \) is given by the equations

\[
y_0 = y_1 = \ldots = y_n = 0,
\]

while \( H_2 \) is given by

\[
x_0 = \ldots = x_{n+1} = 0.
\]

We regard

\[
X \subset H_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}, \quad Y \subset H_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}
\]

as subvarieties of \( \mathbb{P}^{2n+1} \). We form the join \( J(X,Y) \) in \( \mathbb{P}^{2n+1} \).

We first prove that \( J(X,Y) \) has dimension \( \dim X + \dim Y + 1 \). Indeed, any point \( P \) in \( J(X,Y) \) lies on a line \( L \) which intersects both \( X \) and \( Y \) in two points \( Q \) and \( R \). The map

\[
f : J(X,Y) \to X \times Y, P \mapsto (Q,R)
\]

is a well-defined morphism since given \( P \), then \( Q \) and \( R \) are uniquely defined. Indeed, if \( P \in J(X,Y) \) has coordinates \([p_0 : \ldots : p_{2n+1}]\) then \( Q = [p_0 : \ldots : p_n : 0 \ldots : p_n] \) and \( R = [0 : \ldots : 0 : p_{n+1} : \ldots : p_{2n+1}] \), as claimed. The fibers of \( f \) are lines \( QR \), hence they are 1 dimensional. Thus

\[
\dim J(X,Y) = \dim (X \times Y) + 1 = \dim X + \dim Y + 1 \geq n + 1.
\]

Even stronger, by the previous problem, \( J(X,Y) \) is irreducible since it fibers over the irreducible set \( X \times Y \) with equidimensional fibers.

Next, let \( K_i \) be the hyperplane \( x_i - y_i = 0 \) for \( 0 \leq i \leq n \). We claim that

\[
X \cap Y \cong J(X,Y) \cap K_0 \cap K_1 \cap \ldots \cap K_n.
\]

Indeed, any point \( P \in J(X,Y) \) lies on a line \( QR \) with \( Q \in X, R \in Y \), hence

\[
P = \alpha Q + \beta R = [\alpha q : \beta r],
\]
where \( q \) and \( r \) are the homogeneous coordinates of \( Q \) and \( R \) in \( \mathbb{P}^n \). The requirement that

\[
P \in \bigcap_{0 \leq i \leq n} K_i
\]

means that

\[
\alpha q_i = \beta r_i
\]

hence \( Q = R \). This means \( P = Q = R \in X \cap Y \), proving the above equality.

Finally, Intersecting with a hyperplane either keeps the same dimension or cuts the dimension down by 1, hence

\[
\dim(J(X,Y) \cap K_0 \cap \ldots K_n) \geq 0 \implies X \cap Y \neq \emptyset.
\]

\( \square \)

**Problem 5.** (Lines on hypersurfaces.)

(i) Let \( d > 2n - 3 \). Show that a general degree \( d \) hypersurface in \( \mathbb{P}^n \) contains no lines.

(ii) Any cubic surface in \( \mathbb{P}^3 \) contains at least one line.

(iii) Let \( f \) be a degree 4 homogeneous polynomial in 4 variables and let \( Z_f \) be the quartic surface \( f = 0 \) in \( \mathbb{P}^3 \). Show that there is a single polynomial \( \Phi \) in the coefficients of \( f \) which vanishes if and only if the quartic surface \( Z_f \subset \mathbb{P}^3 \) contains a line.

**Answer:** (i) We think of a hypersurface \( X = Z(f) \) as a point in projective space \( \mathbb{P}^N \) for \( N = \binom{n+d}{d} - 1 \), by means of the coefficients \( a_I \) of its defining equation

\[
f = \sum a_I X^I.
\]

We form the incidence correspondence

\[
J = \{(L, X) : L \subset X \} \subset \mathbb{G}(1, n) \times \mathbb{P}^N
\]

and we let

\[
p : J \to \mathbb{G}(1, n), \quad q : J \to \mathbb{P}^N
\]

be the two projections.

We claim that the fibers of \( p \) have dimension \( N - (d + 1) \). Indeed, fix a line \( L \) and study \( p^{-1}(L) \). Without loss of generality, we may assume \( L \) is given by the equations

\[
x_0 = \ldots = x_{n-2} = 0.
\]

If \( X \in p^{-1}(L) \) is given by the polynomial

\[
f = 0,
\]

the requirement \( L \subset X \) means

\[
f(0 : \ldots : 0 : s : t) = 0
\]
for all $s, t$. In particular, the $d+1$ coefficients of $s^t d^{-1}$ for $0 \leq i \leq d$ must vanish:

$$a_{0 \ldots 0, i, d-i} = 0,$$

while the other coefficients are arbitrary. Thus $p^{-1}(L)$ has codimension $d + 1$ in $\mathbb{P}^N$, as claimed. Also the fibers of $p$ are irreducible so $J$ is irreducible as well by Problem 2.

With this understood, we conclude by looking at the fibers of $p$ that

$$\dim J = \dim \mathbb{G}(1, n) + N - (d + 1) = (2n - 2) + N - (d + 1) < N.$$

Therefore, the morphism $q$ is not surjective. In particular, the image $q(J)$ is a proper subvariety of $\mathbb{P}^N$. For hypersurfaces $X$ belonging to the complement $\mathbb{P}^n \setminus q(J)$, the preimage $q^{-1}(X)$ is therefore empty, or in other words, for there are no lines lying on such hypersurfaces.

(ii) We have $d = n = 3$, so that $N = 19$. In this case, the above computation shows $\dim J = N = 19$. It suffices to show $q$ is surjective onto $\mathbb{P}^N$, since then for each cubic $X$ there should be an element in $q^{-1}([X])$ in $J$, thus giving a line $L \subset X$.

The image of $q$ is closed and irreducible in $\mathbb{P}^N$. If $q$ is not surjective, the image is of dimension $N - 1$ or lower. All fibers of $q$ will have dimension at least $N - (N - 1) = 1$. We construct a cubic containing only finitely many lines. For example, we can take

$$X = \{x^3 + y^3 + z^3 + w^3 = 0\}.$$

By symmetry, we may search for lines of the form $x = az + bw, y = cz + dw$ and substituting we find

$$(az + bw)^3 + (cz + dw)^3 + z^3 + w^3 = 0.$$  

This gives

$$a^3 + c^3 + 1 = b^3 + d^3 + 1 = 0, \quad a^2b + c^2d = 0, \quad ab^2 + cd^2 = 0.$$  

We claim that there are finitely many solutions for $a, b, c, d$. If $a = 0$ then it is easy to conclude that $d = 0$ and $b, c$ have to satisfy $b^3 = c^3 = -1$, and the solution set is finite. Assume now that neither $a, b, c, d$ is zero. Then

$$a^2b = -c^2d, ~ ab^2 = -cd^2 \implies \frac{(a^2b)^2}{(ab^2)} = -\frac{(c^2d)^2}{(cd^2)} \implies a^3 = -c^3$$

which contradicts $a^3 + c^3 = -1$.

(iii) In this case, we have $d = 4, n = 3, N = 34$. Let

$$J = \{(L, X) : L \subset X\}.$$  

In this case, the above computation show that $\dim J = N - 1$. We claim that the image $q(J)$ is a codimension 1 subvariety of $\mathbb{P}^N$. We complete the proof letting $\Phi$ be a polynomial cutting out $q(J)$. 


To prove \( q(J) \) is of dimension 33, assume otherwise, namely that the dimension is 32 or lower. By the theorem of dimension of fibers, for all \([X] \in q(J)\), the fiber \( q^{-1}([X]) \) has dimension at least \( 33 - 32 = 1 \). In other words all quartics that contain at least one line in fact contain infinitely many lines. One example is the quartic

\[
x^4 + y^4 + z^4 + w^4 = 0.
\]

By symmetry, we may search for lines of the form \( x = az + bw, y = cz + dw \) and substituting we find

\[
(az + bw)^4 + (cz + dw)^4 + z^4 + w^4 = 0.
\]

This gives

\[
a^4 + c^4 + 1 = b^4 + d^4 + 1 = 0, a^2b^2 + c^2d^2 = 0, a^3b + c^3d = 0, ab^3 + cd^3 = 0.
\]

We claim that there are finitely many solutions for \( a, b, c, d \). If \( a = 0 \) then it is easy to conclude that \( d = 0 \) and \( b, c \) have to satisfy \( b^4 = c^4 = -1 \), and the solution set is finite. Assume now that neither \( a, b, c, d \) is zero. Then

\[
a^3b = -c^3d, \quad ab^3 = -cd^3 \implies (a/b)^2 = (c/d)^2
\]

and in addition \( (ab)^2 = -(cd)^2 \) so multiplying we find \( a^4 = -c^4 \) which contradicts \( a^4 + c^4 = -1 \).

\[\square\]