Problem 1. Given four general lines in $\mathbb{P}^3$, show that there are exactly 2 lines which intersect all four of them.

Answer: Recall that the space of lines in $\mathbb{P}^3$ is parametrized by the Grassmannian $G = G(1, 3)$ which can be realized as a quadric in $\mathbb{P}^5$ via the Plucker embedding. For each line $L_i$ define

$$X_i = \{ M \text{ line in } \mathbb{P}^3 : M \cap L_i \neq \emptyset \} \subset G(1, 3) \subset \mathbb{P}^5.$$

We claim that

$$X_i = H_i \cap G$$

for a hyperplane $H_i$ in $\mathbb{P}^5$. Indeed, working in Plucker coordinates, assume that $L = L_i$ has coordinates $l_{ij}$ and $M$ has Plucker coordinates $m_{kl}$. If these are calculated with respect to points $A, B$ over $L$ and points $C, D$ on $M$ then we have

$$a \wedge b = \sum_{ij} l_{ij} e_i \wedge e_j$$

$$c \wedge d = \sum_{kl} m_{kl} e_k \wedge e_l.$$

The requirement that $L$ and $M$ meet is equivalent to

$$a \wedge b \wedge c \wedge d = 0$$

since the vector space spanned by $a, b, c, d$ is 3 dimensional in this case. This gives

$$\left( \sum_{ij} l_{ij} e_i \wedge e_j \right) \wedge \left( \sum_{kl} m_{kl} e_k \wedge e_l \right) = 0$$

which gives

$$l_{12}m_{34} - l_{13}m_{24} + l_{14}m_{32} + l_{23}m_{14} - l_{24}m_{34} + l_{34}m_{12} = 0.$$

This is clearly a linear equation in the variables $m_{kl}$ for each fixed $l_{ij}$.

Now, the lines $M$ that intersect $L_1, L_2, L_3, L_4$ are found as the intersection points

$$X_1 \cap X_2 \cap X_3 \cap X_4 \subset G(1, 3).$$

In other words, these points correspond to

$$H_1 \cap H_2 \cap H_3 \cap H_4 \cap G(1, 3) \subset \mathbb{P}^5.$$

We claim that this intersection consists of 2 points in general.

We claim first that the intersection $H_1 \cap H_2 \cap H_3 \cap H_4$ is a line $\ell$ in $\mathbb{P}^5$ in general. In any case, the intersection is given as the null space of the $4 \times 6$ matrix of coefficients describing the hyperplanes $H_i$. In general, this null space is 1 dimensional, so the intersection is a line, but it can also be that the null space has dimension 2 or higher. This condition is
described as the rank of the matrix being 4 or less – in turn this is given by the vanishing of the $4 \times 4$ minors, so it is a closed subset $Z$ in the space $G \times G \times G \times G$. We assume $(L_1, L_2, L_3, L_4)$ are chosen away from $Z$.

Next, if $\ell$ is the intersection line, we claim it intersects the quadric $G$ in $\mathbb{P}^5$ in 2 points. Indeed, we may assume that after a change of coordinates, this line is given by $x_2 = x_3 = x_4 = x_5 = 0$. The quadric $G$ will be given by $\sum a_{ij} x_i x_j$ and the intersection of the line $\ell$ is obtained by solving

$$a_{00} x_0^2 + a_{11} x_1^2 + a_{01} x_0 x_1 = 0$$

which has exactly two solutions. The only exceptions correspond to

$$a_{01} = 4a_{00} a_{11}$$

which corresponds to one solution, or the case $a_{00} = a_{01} = a_{11} = 0$ which corresponds to infinitely many solutions. These are closed conditions determining a closed set $W$ as one can check.

Setting $U = G \setminus (Z \cup W)$ we obtain that for $(L_1, L_2, L_3, L_4)$ in $U$, there are exactly 2 lines intersecting $L_i$. Then $U$ is dense in $G \times G \times G \times G$ if nonempty. To show nonemptyness, we can pick 4 lines

$L_1 = \{x_0 = x_1 = 0\}, L_2 = \{x_0 = x_2 = 0\}, L_3 = \{x_0 + x_1 = x_2 + x_3 = 0\}$

$L_4 = \{x_0 + 2x_1 = x_2 + 2x_3 = 0\}$.

We claim this quadruple lies in $U$. Indeed, one can easily run the argument above to find the equations of the hyperplanes $H_1, H_2, H_3, H_4$ above in terms of the Plucker coordinates. We obtain

$$m_{01} = 0, m_{02} = 0, m_{13} + m_{03} + m_{12} + m_{02} = 0, 4m_{13} + 2m_{03} + 2m_{12} + m_{02} = 0.$$  

We also have

$$m_{01} m_{23} - m_{02} m_{13} + m_{03} m_{12} = 0$$

for the equation of the quadric. These equations only have 2 common solutions as one checks immediately. □

**Problem 2.** Show that the Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}(n+1)(m+1)-1$$

has degree $\binom{n+m}{n}$.

**Answer:** Let $\Sigma_{m,n}$ be the image of the Segre embedding. Degree $\ell$ homogeneous polynomials on $\mathbb{P}(n+1)(m+1)-1$ restrict to $\Sigma_{m,n}$ as polynomials in the variables $x_i$ on $\mathbb{P}^n$ and $y_j$ on $\mathbb{P}^m$, bihomogeneous of degree $\ell$. The dimension of $S(\Sigma_{m,n})^{(\ell)}$ then equals

$$\binom{\ell + n}{n} \binom{\ell + m}{m}.$$
Expanding, we have
\[
\binom{\ell + n}{n} \binom{\ell + m}{m} = \frac{1}{n!m!} \cdot \ell^{n+m} + \text{l.o.t.}
\]
which shows that the degree of \( \Sigma_{m,n} \) equals
\[
\frac{(n+m)!}{n!m!} = \binom{n+m}{n}.
\]

\( \square \)

**Problem 3.** Let \( X \subset \mathbb{P}^n \) be a projective scheme with Hilbert polynomial \( \chi_X \). Define the arithmetic genus of \( X \) to be
\[
p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1).
\]

(i) Show that the genus of \( \mathbb{P}^n \) is zero.

(ii) If \( X \) is a hypersurface of degree \( d \) in \( \mathbb{P}^n \), show that \( p_a(X) = \binom{d-1}{n} \). In particular, a cubic in \( \mathbb{P}^2 \) has genus 1.

(iii) If \( X \) is a complete intersection of two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) then
\[
p_a(X) = \frac{1}{2}ab(a+b-4) + 1.
\]

In particular, intersection of two quadrics in \( \mathbb{P}^3 \) has genus 1.

**Answer:**

(i) We calculated the Hilbert polynomial of \( \mathbb{P}^n \) to be \( \chi(\ell) = \binom{\ell+n}{n} \). This yields immediately that \( p_a(\mathbb{P}^n) = 0 \).

(ii) We calculated the Hilbert polynomial of a degree \( d \) hypersurface to be
\[
\chi(\ell) = \binom{n+\ell}{n} - \binom{n+\ell-d}{n}.
\]
This yields
\[
p_a(X) = (-1)^{n-1} \left( 1 - \binom{n-d}{n} - 1 \right) = (-1)^n \binom{n-d}{n} = \binom{d-1}{n}.
\]

(iii) We claim that the Hilbert polynomial of the complete intersection equals
\[
\chi(\ell) = \binom{\ell+3}{3} - \binom{\ell+3-a}{3} - \binom{\ell+3-b}{b} + \binom{\ell+3-a-b}{3}.
\]
Then, we find
\[
\chi(0) = 1 - \binom{3-a}{a} - \binom{3-b}{b} + \binom{3-a-b}{3}
\]
which yields the answer.

The claim about the Hilbert polynomial is justified as follows. Let \( f \) and \( g \) be the equations of the two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) whose intersection is \( X \). There is an exact sequence
\[
0 \to S(\mathbb{P}^3)^{(\ell-a-b)} \to S(\mathbb{P}^3)^{(\ell-a)} \oplus S(\mathbb{P}^3)^{(\ell-b)} \to S(\mathbb{P}^3)^{(\ell)} \to S(X)^{(\ell)} \to 0
\]
where the first two maps are given by
\[ P \mapsto (gP, fP) \]
and
\[ (P, Q) \mapsto fP - gQ \]
and the last map is the restriction. We conclude by considering dimensions.

\[ \square \]

**Problem 4.** Let \( X \) be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree \( d \) and codimension \( c \) in \( \mathbb{P}^n \).

(i) (Intersecting \( X \) with hyperplanes to cut down the dimension), show inductively that
\[ d \geq c + 1. \]

(ii) Both cases are particular examples of the Veronese embedding whose degree we calculate below.

Consider Veronese embedding
\[ v_d : \mathbb{P}^n \to \mathbb{P}^N \]
constructed from degree \( d \) monomials. We claim that the image \( V_d \) has degree \( d^n \). Indeed, degree \( \ell \) polynomials in \( N + 1 \) variables become, after restricting to \( V_d \), polynomials of degree \( d \ell \) on \( \mathbb{P}^n \). Hence the Hilbert function of \( V_d \) equals
\[ \chi(\ell) = \binom{d\ell + n}{n} = d^n \frac{\ell^n}{n!} + \text{l.o.t}, \]
confirming the claim.
Now, it is easy to see that
\[ d^n = \text{codim } V_d + 1 = \binom{d+n}{n} - n \]
holds for \( n = 1 \) or for \( d = n = 2 \).

(iii) After passing to a smaller projective space, we may assume that \( X \) is nondegenerate. Degree \( d = 2 \) forces \( c = 1 \) hence \( X \) is isomorphic to a projective quadric.

\[ \square \]