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Projections

- $X \subset \mathbb{P}^n$, $X \neq \mathbb{P}^n$
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First properties

**Lemma**

*Let* $X$ *be a variety of dimension* $n$.

(i) *if* $Y \subset X$ *is closed,*

\[ \dim Y < \dim X \]
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Let $X$ be a variety of dimension $n$.

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Adjoin $X_{i+1}, \ldots, X_n$ to get a chain in $X$.
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\[ \dim X_i + (n - i) > n. \]
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Contradiction!
Surjective maps

Lemma
Let $f : X \to Y$ be a surjective morphism of projective varieties. Every longest chain in $Y$:

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = Y$$

can be lifted to a chain in $X$:

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$$\dim X \geq \dim Y.$$
Proof:

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- **Induct** on \( \dim Y \).
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- **irreducible components**

\[
Z = f^{-1}(Y_{n-1}) = Z_1 \cup \ldots \cup Z_r
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Proof:

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- $Z$ is closed in $X$.
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Let $X \subsetneq \mathbb{P}^n$ be irreducible, and $p \not\in X$, $p = [0 : \ldots : 0 : 1]$. 
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Let $X \subseteq \mathbb{P}^n$ be irreducible, and $p \notin X$, $p = [0 : \ldots : 0 : 1]$.
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**Proposition (Noether normalization)**
Let $X \subseteq \mathbb{P}^n$ be irreducible, and $p \notin X$, $p = [0 : \ldots : 0 : 1]$.
Let $f \in k[x_0, \ldots, x_n]$ homogeneous. There exists $D > 0$ and $a_1, \ldots, a_D \in k[x_0, \ldots, x_{n-1}]$ homogeneous such that

$$f^D + a_1 f^{D-1} + \ldots + a_D = 0 \text{ on } X.$$
Expanded solution:

Pick $f$ homogeneous, $f \in I(Y) \setminus I(X)$.

$\Rightarrow$ Take $f D + a_1 f D - 1 + \ldots + a_D = 0$ on $X$ with $D$ minimal.

$\Rightarrow$ For $y \in Y \subset X$ $f(y) D + a_1(\pi(y)) f(y) D - 1 + \ldots + a_D(\pi(y)) = 0$

$\Rightarrow$ Since $f(y) = 0$ for $y \in Y$, $a_D(\pi(Y)) = 0 \Rightarrow a_D = 0$ on $\pi(Y)$.

$\Rightarrow$ If $a_D = 0$ on $\pi(X)$ then $f D + a_1 f D - 1 + \ldots + a_D = 0$ on $X$.

$\Rightarrow$ Since $S(X)$ is integral domain, and $f \neq 0$ in $S(X)$. 
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contradicting minimality of \( D \).
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Proposition

\[ \dim \mathbb{P}^n = n. \]

Proof:

- Let \( f(n) = \dim \mathbb{P}^n \)
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Proof:

Let $$f(n) = \dim \mathbb{P}^n$$

Claim: no gaps in the values of $$f$$.

Hence $$f(n) = n$$.

Equivalently: If $$d$$ is a value of $$f$$, then all $$0 \leq i < d$$ are assumed by $$f$$. 

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \ldots \subset \mathbb{P}^n \subset \ldots$$
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- \( \mathbb{P}^0 \subsetneq \mathbb{P}^1 \subsetneq \ldots \subsetneq \mathbb{P}^n \subsetneq \ldots \)

\[ \implies f(0) < f(1) < \ldots < f(n) < \ldots \]
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Next: proof and discussion of Noether normalization.
Proposition (Noether normalization)

Let $X \subseteq \mathbb{P}^n$ be irreducible, and $p \not\in X$, $p = [0 : \ldots : 0 : 1]$. 

Remark: ▶ The fibers of $\pi: X \to \mathbb{P}^n - 1$ have $\leq D$ points.
▶ A morphism with finite fibers will be called quasi-finite.
Proposition (Noether normalization)
Let $X \subset \mathbb{P}^n$ be irreducible, and $p \notin X$, $p = [0 : \ldots : 0 : 1]$.
Let $f \in k[x_0, \ldots, x_n]$ homogeneous.
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$\Rightarrow d = \deg f$. Define

$$\Phi : X \to \mathbb{P}^n, \quad \Phi(x) = [x_0^d : \ldots : x_{n-1}^d : f].$$
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- $\Phi(X)$ is projective cut out by $F_1 = \ldots = F_r = 0$.

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Discuss affine varieties and arbitrary varieties

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