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Composition of blowups!!
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Composition of **blowups**!!

- **Further motivation:** If $f : X \dasharrow Y$ we use **blowups** to extend to regular

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Blowups – outline:

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Terminology: strict transform, exceptional set
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Blowup of the plane (simplified)

- **Blowup morphism**

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\tilde{C} \cap \ell = \{ f^{in}(1, \nu) = 0 \} = \left\{ \prod_{k} (\alpha_k + \beta_k \nu) = 0 \right\} = \left\{ -\frac{\alpha_k}{\beta_k} \right\}
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- \( \pi : \tilde{C} \to C \) birational
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\[ \widetilde{C} \cap \widetilde{D} = \emptyset \implies \text{strict transforms separated} \]
Figure: Blowing up the plane
Example: Node:

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One more blowup resolves the singularity.
Blowup of $\mathbb{A}^n$

$p = (0, \ldots, 0),$

$\widetilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}, \quad \widetilde{\mathbb{A}}^n = \{x_i y_j - x_j y_i = 0\}, \quad x \in \mathbb{A}^n, \ y \in \mathbb{P}^{n-1}$
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Affine cover

\[ \widetilde{\mathbb{A}}^n = \bigcup_{i} U_i, \quad U_i = \{(x, y) : y_i \neq 0\} \]
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Example: When \( n = 2 \),

\[ U_1 \simeq \mathbb{A}^2, \quad (x_1, y_2) \mapsto (x_1, x_1y_2) \]

studied previously.
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General construction

- Data

\[ X \subset \mathbb{A}^n, \; Y = \{f_1 = \ldots = f_r = 0\} \subset X, \; U = X \setminus Y \]
General construction

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- **Graph**

\[ \Gamma_f \subset U \times \mathbb{P}^{r-1} \]
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\[ \Gamma_f \cong U \implies \Gamma_f \text{ irreducible} \implies \tilde{X} \text{ irreducible} \]
General construction

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Terminology: Exceptional hypersurface

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Equations for the blowup

Lemma

\[ \tilde{X} \subset \{(x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x)\} \]

Proof: Over \( U \):

\[ \Gamma_f = \{(x, y) : y = f(x)\} = \{y_i f_j(x) = y_j f_i(x)\} \]

Remark: This recovers \( \tilde{\mathbb{A}}^n \) for \( f_i(x) = x_i, \ X = \mathbb{A}^n, \ Y = \{0\} \).

BEWARE!!! Equality may not hold above.
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Thus the same equations hold over \( \tilde{X} = \Gamma_f \).
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Two blowups

$$\tilde{X} \subset X \times \mathbb{P}^{r-1}, \quad \tilde{X}' \subset X \times \mathbb{P}^{s-1}.$$
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Several methods:

- $X$ variety, $Y \subset X$ closed, $X = \bigcup U_i$. 

Construct $\tilde{X}$: blowup $U_i$ at $Y \cap U_i$ and glue.
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Example: del Pezzo surfaces

Blowups of $\mathbb{P}^2$ at $n \leq 8$ general points.
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**Exercise:** Blowup of $\mathbb{P}^2$ at 1 point $\simeq$ blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 2 points.
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Figure: Pasquale del Pezzo
Example: Cremona transformation

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