Math 203A – December 5, 2018

Office hour: 3:45-5:00
Last time

- $i \subset S = k[x_0, \ldots, x_n]$ homogeneous
Last time

- $i \subset S = k[x_0, \ldots, x_n]$ homogeneous
- Hilbert function

\[ h_i(d) = \dim_k (S/i)^{(d)} \]
Last time

- \( i \subset S = k[x_0, \ldots, x_n] \) homogeneous
- Hilbert function

\[
h_i(d) = \dim_k(S/i)^{(d)}
\]

- condition (*): \( \deg f = e \):

\[
h_{i+(f)}(d) = h_i(d) - h_i(d - e).
\]
Last time

- $i \subset S = k[x_0, \ldots, x_n]$ homogeneous
- Hilbert function

\[ h_i(d) = \dim_k(S/i)^{(d)} \]

- condition (*): $\deg f = e$:

\[ h_{i+(f)}(d) = h_i(d) - h_i(d - e). \]

- Condition (*):

\[ fg \in i \implies g \in i \text{ for } \deg g \geq d_0. \]
Hilbert polynomial

**Theorem**

Let $i$ homogeneous."
Hilbert polynomial

Theorem

Let \( i \) homogeneous. There exists a polynomial \( \chi_i \in \mathbb{Q}[d] \) such that

\[
h_i = \chi_i \text{ for } d \gg 0.
\]
Hilbert polynomial

Theorem

Let $i$ homogeneous. There exists a polynomial $\chi_i \in \mathbb{Q}[d]$ such that

$$h_i = \chi_i \text{ for } d >> 0.$$ 

Furthermore

$$\chi_i = \frac{d^m}{m!} \cdot \text{integer} + \ldots$$
Theorem

Let $i$ homogeneous. There exists a polynomial $\chi_i \in \mathbb{Q}[d]$ such that

$$h_i = \chi_i \text{ for } d >> 0.$$ 

Furthermore

$$\chi_i = \frac{d^m}{m!} \cdot \text{integer} + \ldots$$

where

$\blacktriangleright \ m = \dim \mathbb{Z}_p(i)$
Hilbert polynomial

**Theorem**

*Let* \( i \) *homogeneous. There exists a polynomial* \( \chi_i \in \mathbb{Q}[d] \) *such that*

\[
h_i = \chi_i \text{ for } d \gg 0.
\]

*Furthermore*

\[
\chi_i = \frac{d^m}{m!} \cdot \text{integer} + \ldots
\]

*where*

- \( m = \text{dim} \ Z_p(i) \)
- \( \text{integer} \) \( := \deg i / \mathbb{P}^n. \)
Hilbert polynomial

Theorem

Let $i$ homogeneous. There exists a polynomial $\chi_i \in \mathbb{Q}[d]$ such that

$$h_i = \chi_i \text{ for } d >> 0.$$ 

Furthermore

$$\chi_i = \frac{d^m}{m!} \cdot \text{integer} + \ldots$$

where

- $m = \dim Z_p(i)$
- $\text{integer} := \deg i / \mathbb{P}^n.$

For $X \subset \mathbb{P}^n$

$$\chi_X(d) = \frac{d^m}{m!} \cdot \deg X + \ldots, m = \dim X.$$
Question: Remaining coefficients?
**Question:** Remaining coefficients?

**Answer:** Math 203C, Chern classes, Hirzerbruch-Riemann-Roch for $X$ smooth
Question: Remaining coefficients?

Answer: Math 203C, Chern classes, Hirzerbruch-Riemann-Roch for $X$ smooth

Today: Constant term.
Question: Remaining coefficients?

Answer: Math 203C, Chern classes, Hirzerbruch-Riemann-Roch for \( X \) smooth.

Today: Constant term.

Definition
For projective \( X \), the arithmetic genus

\[
p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1)
\]
**Question:** Remaining coefficients?

**Answer:** Math 203C, Chern classes, Hirzerbruch-Riemann-Roch for $X$ smooth

**Today:** Constant term.

**Definition**
For projective $X$, the *arithmetic genus* $p_a(X)$

$$p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1)$$

**Hard:** For $X$ smooth curves, coincides with the *topological genus*. 
**Question:** Remaining coefficients?

**Answer:** Math 203C, Chern classes, Hirzerbruch-Riemann-Roch for $X$ smooth

**Today:** Constant term.

**Definition**
For projective $X$, the arithmetic genus

$$p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1)$$

**Hard:** For $X$ smooth curves, coincides with the topological genus.
Proof: Induct on $m = \dim Z_p(i)$.
Proof: Induct on \( m = \dim Z_p(i) \).

Base case: \( \dim Z_p(i) = \emptyset \), \( m = -\infty \) and \( \chi_i = 0 \) (previously).
Proof: Induct on $m = \dim Z_p(i)$.

Base case: $\dim Z_p(i) = \emptyset$, $m = -\infty$ and $\chi_i = 0$ (previously).

Inductive step: Take $f$ linear to satisfy (*).
Proof: Induct on $m = \dim \mathbb{Z}_p(i)$.

Base case: $\dim \mathbb{Z}_p(i) = \emptyset$, $m = -\infty$ and $\chi_i = 0$ (previously).

Inductive step: Take $f$ linear to satisfy (*)..

$f$ doesn’t vanish identically on components of $X$:

$$\dim \mathbb{Z}_p(i + (f)) = m - 1.$$
Proof: Induct on $m = \dim Z_p(i)$.

Base case: $\dim Z_p(i) = \emptyset$, $m = -\infty$ and $\chi_i = 0$ (previously).

Inductive step: Take $f$ linear to satisfy (*).

$f$ doesn't vanish identically on components of $X$:

$$\dim Z_p(i + (f)) = m - 1.$$ 

$$h_{i+(f)}(d) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t., } \alpha \in \mathbb{Z}, \quad d \geq d_0$$
Proof: Induct on \( m = \dim Z_p(i) \).

Base case: \( \dim Z_p(i) = \emptyset, \ m = -\infty \) and \( \chi_i = 0 \) (previously).

Inductive step: Take \( f \) linear to satisfy (*).

\( f \) doesn’t vanish identically on components of \( X \):

\[
\dim Z_p(i + (f)) = m - 1.
\]

\[
h_{i+(f)}(d) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t., } \alpha \in \mathbb{Z}, \ d \geq d_0
\]

WTS: \( h_i(d) - h_i(d - 1) = \text{polynomial} \iff h_i \text{ polynomial} \)
Proof: Induct on $m = \dim Z_p(i)$.

Base case: $\dim Z_p(i) = \emptyset$, $m = -\infty$ and $\chi_i = 0$ (previously).

Inductive step: Take $f$ linear to satisfy (*).

$f$ doesn’t vanish identically on components of $X$:

$$\dim Z_p(i + (f)) = m - 1.$$

$$h_{i+(f)}(d) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t., } \alpha \in \mathbb{Z}, \ d \geq d_0$$

WTS: $h_i(d) - h_i(d - 1) = \text{polynomial} \implies h_i \text{ polynomial}$
Claim:

\[(d_0), (d_1), \ldots, (d_k)\]

form a basis for the polynomials of degree \( \leq k \).
Claim: 

\((d_0), (d_1), \ldots, (d_k)\)

form a basis for the polynomials of degree \(\leq k\).

Proof: Suffices to check independence.
Claim:

\[(d_0, d_1, \ldots, d_k)\]

form a basis for the polynomials of degree \(\leq k\).

Proof: Suffices to check independence.

\[
\sum_{i=0}^{k} a_i \binom{d}{i} = 0
\]
Claim: \((d_0), (d_1), \ldots, (d_k)\) form a basis for the polynomials of degree \(\leq k\).

Proof: Suffices to check independence.

\[
\sum_{i=0}^{k} a_i \binom{d}{i} = 0 \implies a_0 = 0 \text{ for } d = 0
\]
Claim:

\[
\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{k}
\]

form a basis for the polynomials of degree \( \leq k \).

Proof: Suffices to check independence.

\[
\sum_{i=0}^{k} a_i \binom{d}{i} = 0 \implies a_0 = 0 \text{ for } d = 0
\]

\[
\implies \sum_{i=1}^{k} \left( a_i \cdot \frac{d}{i} \right) \cdot \binom{d-1}{i-1} = 0
\]
Claim: \((d^0), (d^1), \ldots, (d^k)\) form a basis for the polynomials of degree \(\leq k\).

Proof: Suffices to check independence.

\[
\sum_{i=0}^{k} a_i \binom{d}{i} = 0 \implies a_0 = 0 \text{ for } d = 0
\]

\[
\implies \sum_{i=1}^{k} \left( a_i \cdot \frac{d}{i} \right) \cdot \binom{d-1}{i-1} = 0
\]

\[
\implies a_i \cdot \frac{d}{i} = 0 \text{ (induction)}
\]
Claim:

\[
\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{k}
\]

form a basis for the polynomials of degree $\leq k$.

Proof: Suffices to check independence.

\[
\sum_{i=0}^{k} a_i \binom{d}{i} = 0 \implies a_0 = 0 \text{ for } d = 0
\]

\[
\implies \sum_{i=1}^{k} \left( a_i \cdot \frac{d}{i} \right) \cdot \binom{d-1}{i-1} = 0
\]

\[
\implies a_i \cdot \frac{d}{i} = 0 \text{ (induction)} \implies a_i = 0
\]
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} \]
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

- **highest term** \( c_{m-1} = \alpha \in \mathbb{Z} \)
\[ h_{i+(f)} = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

- **highest term** \( c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d - 1) = h_{i+(f)}(d) \]
\[ h_{i+(f)} = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

- highest term \( c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d-1) = h_{i+(f)}(d) = \sum c_i \binom{d}{i} \text{ for } d >> 0 \]
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

*highest term* \( c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d - 1) = h_{i+}(f)(d) = \sum c_i \binom{d}{i} \text{ for } d >> 0 \]

Claim:

\[ h_i(d) = c + \sum_{i=0}^{m-1} c_i \binom{d + 1}{i + 1} \text{ for } d >> 0 \]
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

\( \text{highest term } c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d-1) = h_{i+}(f)(d) = \sum c_i \binom{d}{i} \text{ for } d \gg 0 \]

Claim:

\[ h_i(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} \text{ for } d \gg 0 \]

Then:

\( h_i \) is polynomial of degree \( m \)
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

- highest term \( c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d - 1) = h_{i+}(f)(d) = \sum c_i \binom{d}{i} \text{ for } d \gg 0 \]

Claim:
\[ h_i(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} \text{ for } d \gg 0 \]

Then:
- \( h_i \) is polynomial of degree \( m \)
- highest term \( c_{m-1} = \alpha \in \mathbb{Z} \)
\[ h_{i+}(f) = \frac{d^{m-1}}{(m-1)!} \cdot \alpha + \text{l. o. t.} = \sum_{i=0}^{m-1} c_i \binom{d}{i} \]

- highest term \( c_{m-1} = \alpha \in \mathbb{Z} \)

\[ h_i(d) - h_i(d-1) = h_{i+}(f)(d) = \sum c_i \binom{d}{i} \text{ for } d >> 0 \]

Claim:
\[ h_i(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} \text{ for } d >> 0 \]

Then:
- \( h_i \) is polynomial of degree \( m \)
- highest term \( c_{m-1} = \alpha \in \mathbb{Z} \)
Proof of claim: Induct on $d \geq d_0$. 
Proof of claim: Induct on \( d \geq d_0 \).

Pick \( c \) to arrange equality at \( d = d_0 \).
Proof of claim: Induct on $d \geq d_0$.

Pick $c$ to arrange equality at $d = d_0$.

Inductive step:

$$h_i(d) - h_i(d - 1) = \sum c_i \binom{d}{i}$$
Proof of claim: Induct on $d \geq d_0$.

Pick $c$ to arrange equality at $d = d_0$.

Inductive step:

$$h_i(d) - h_i(d - 1) = \sum c_i \binom{d}{i}$$

$$h_i(d) = \left( \sum c_i \binom{d}{i} \right) + \left( c + \sum c_i \binom{d}{i+1} \right)$$

$$= \quad$$
Proof of claim: Induct on $d \geq d_0$.

Pick $c$ to arrange equality at $d = d_0$.

Inductive step:

\[
\begin{align*}
  h_i(d) - h_i(d - 1) &= \sum c_i \binom{d}{i} \\
  h_i(d) &= \left( \sum c_i \binom{d}{i} \right) + \left( c + \sum c_i \binom{d}{i+1} \right) \\
  &= c + \sum c_i \binom{d+1}{i+1}
\end{align*}
\]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} \]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} = \binom{d + n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} = \binom{d + n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]

\[ \implies \deg \mathbb{P}^n = 1 \]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} \]
\[ = \binom{d + n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]
\[ \implies \deg \mathbb{P}^n = 1 \]

Example: \( X \subset \mathbb{P}^n, H \subset \mathbb{P}^n \). Proof shows

\[ \deg X = \deg X \cap H. \]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^<(d) \]
\[ = \binom{d + n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]
\[ \implies \deg \mathbb{P}^n = 1 \]

Example: \( X \subset \mathbb{P}^n, H \subset \mathbb{P}^n \). Proof shows

\[ \deg X = \deg X \cap H. \]

Example: \( \dim X = \dim Y = m, \dim(X \cap Y) < m \)
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} = \binom{d + n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]

\[ \implies \deg \mathbb{P}^n = 1 \]

Example: \( X \subset \mathbb{P}^n, H \subset \mathbb{P}^n \). Proof shows

\[ \deg X = \deg X \cap H. \]

Example: \( \dim X = \dim Y = m, \dim(X \cap Y) < m \)

\[ h_X + h_Y = h_{X \cap Y} + h_{X \cup Y} \]
Example:

\[ h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]^{(d)} = \binom{d+n}{n} = \frac{d^n}{n!} \cdot 1 + \ldots \]

\[ \implies \deg \mathbb{P}^n = 1 \]

Example: \( X \subset \mathbb{P}^n, H \subset \mathbb{P}^n \). Proof shows

\[ \deg X = \deg X \cap H. \]

Example: \( \dim X = \dim Y = m, \dim(X \cap Y) < m \)

\[ h_X + h_Y = h_{X \cap Y} + h_{X \cup Y} \]

Highest term

\[ \deg X + \deg Y = \deg(X \cup Y). \]
Example: \( i = (f), \deg f = e \)
Example: i = (f), deg f = e

\[ 0 \rightarrow S^{(d-e)} \rightarrow S^{(d)} \rightarrow (S/(f))^{(d)} \rightarrow 0 \]

\[ \phi \mapsto \phi \cdot f \]
Example: \( i = (f), \ deg f = e \)

\[
0 \rightarrow S^{(d-e)} \rightarrow S^{(d)} \rightarrow (S/(f))^{(d)} \rightarrow 0
\]

\[
\phi \mapsto \phi \cdot f
\]

Take dimensions:

\[
h_{\chi}(d) = \binom{d + n}{n} - \binom{d - e + n}{n}
\]
Example: \( i = (f), \ \text{deg} \ f = e \)

\[
0 \to S^{(d-e)} \to S^{(d)} \to (S/(f))^{(d)} \to 0
\]
\[
\phi \mapsto \phi \cdot f
\]

Take dimensions:

\[
h_X(d) = \binom{d+n}{n} - \binom{d-e+n}{n}
\]
\[
= \frac{d^{n-1}}{(n-1)!} \cdot e + \text{l. o. t}
\]
Example: \( i = (f) \), \( \deg f = e \)

\[
0 \to S^{(d-e)} \to S^{(d)} \to (S/(f))^{(d)} \to 0
\]

\( \phi \mapsto \phi \cdot f \)

Take dimensions:

\[
h_X(d) = \left( \binom{d+n}{n} - \binom{d-e+n}{n} \right)
\]

\[
= \frac{d^{n-1}}{(n-1)!} \cdot e + \text{l. o. t}
\]

\( \implies \deg X = e \)

Arithmetic genus

\[
p_a(X) = (-1)^{n-1} \left( 1 - \binom{-e+n}{n} - 1 \right) = \binom{e-1}{n}
\]
Example: \( i = (f), \ deg f = e \)

\[
0 \to S^{(d-e)} \to S^{(d)} \to (S/(f))^{(d)} \to 0
\]

\[
\phi \mapsto \phi \cdot f
\]

Take dimensions:

\[
h_X(d) = \binom{d+n}{n} - \binom{d-e+n}{n}
\]

\[
= \frac{d^{n-1}}{(n-1)!} \cdot e + \text{l. o. t}
\]

\[
\implies \deg X = e
\]

Arithmetic genus

\[
p_a(X) = (-1)^{n-1} \left( 1 - \binom{-e+n}{n} - 1 \right) = \binom{e-1}{n}
\]

Example: \( X \) degree \( e \) plane curve

\[
p_a(X) = \frac{(e-1)(e-2)}{2}.
\]
Example: $i = (f), \ deg f = e$

$$0 \to S^{(d-e)} \to S^{(d)} \to (S/(f))^{(d)} \to 0$$

$$\phi \mapsto \phi \cdot f$$

Take dimensions:

$$h_X(d) = \binom{d + n}{n} - \binom{d - e + n}{n}$$

$$= \frac{d^{n-1}}{(n-1)!} \cdot e + l. o. t$$

$$\implies \deg X = e$$

Arithmetic genus

$$p_a(X) = (-1)^{n-1} \left(1 - \binom{-e + n}{n} - 1\right) = \binom{e - 1}{n}$$

Example: $X$ degree $e$ plane curve

$$p_a(X) = \frac{(e - 1)(e - 2)}{2}.$$
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$. 
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

$$\deg X = r \implies \deg i = r$$
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

$$\deg X = r \implies \deg i = r$$

Assume

$$Z(j) = X = r \text{ distinct points}$$
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = l(X)$.

$$\text{deg } X = r \implies \text{deg } i = r$$

Assume

$Z(j) = X = r$ distinct points

$$Z(j) = Z(i) \implies \sqrt{j} = i$$
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

\[
\text{deg} \ X = r \implies \text{deg} \ i = r
\]

Assume

\[
Z(j) = X = r \text{ distinct points}
\]

\[
Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i
\]
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

\[ \deg X = r \implies \deg i = r \]

Assume

\[ Z(j) = X = r \text{ distinct points} \]

\[ Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i \]

\[ \implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)} \text{ surjective} \]
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

$$\deg X = r \implies \deg i = r$$

Assume

$Z(j) = X = r$ distinct points

$Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i$

$\implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)}$ surjective $\implies h_j \geq r$
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

$$\text{deg } X = r \implies \text{deg } i = r$$

Assume

$$Z(j) = X = r \text{ distinct points}$$

$$Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i$$

$$\implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)} \text{ surjective} \implies h_j \geq r \implies \text{deg } j \geq r$$
Aside (“zero diml schemes”)

If \( X \) consists of \( r \) distinct points, \( i = I(X) \).

\[
deg X = r \implies deg i = r
\]

Assume

\[
Z(j) = X = r \text{ distinct points}
\]

\[
Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i
\]

\[
\implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)} \text{ surjective} \implies h_j \geq r \implies deg j \geq r
\]

deg j counts the \( r \) points with multiplicity
Aside (“zero diml schemes”)

If $X$ consists of $r$ distinct points, $i = I(X)$.

\[ \deg X = r \implies \deg i = r \]

Assume

\[ Z(j) = X = r \text{ distinct points} \]

\[ Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i \]

\[ \implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)} \text{ surjective} \implies h_j \geq r \implies \deg j \geq r \]

deg $j$ counts the $r$ points with multiplicity

\[ \cdot j = (x, y^n) \implies h_j = \dim k[x, y]/(x, y^n) = n \]
Aside ("zero diml schemes")

If $X$ consists of $r$ distinct points, $i = I(X)$.

$$\text{deg } X = r \implies \text{deg } i = r$$

Assume

$$Z(j) = X = r \text{ distinct points}$$

$$Z(j) = Z(i) \implies \sqrt{j} = i \implies j \subset i$$

$$\implies (S/j)^{(d)} \hookrightarrow (S/i)^{(d)} \text{ surjective} \implies h_j \geq r \implies \text{deg } j \geq r$$

deg $j$ counts the $r$ points with multiplicity

$\blacktriangleright$ $j = (x, y^n) \implies h_j = \dim k[x, y]/(x, y^n) = n$
Global Bezout’s theorem

Theorem

$X \subset \mathbb{P}^n$, $f$ not vanishing identically on any component of $X$. 
Global Bezout’s theorem

Theorem

$X \subset \mathbb{P}^n$, $f$ not vanishing identically on any component of $X$.

$$\deg I(X) + (f) = \deg X \cdot \deg f$$
Global Bezout’s theorem

**Theorem**

$X \subset \mathbb{P}^n$, $f$ not *vanishing identically* on any component of $X$.

$$\deg I(X) + (f) = \deg X \cdot \deg f$$
Proof:

\[ h_{l(f)}(d) = h_X(d) - h_X(d - e) \]
Proof:

\[ h_{I(X)+f}(d) = h_X(d) - h_X(d-e) \]
\[ = \frac{d^m}{m!} \cdot \alpha + \frac{d^{m-1}}{(m-1)!} \cdot \beta + \ldots \]
Proof:

\[ h_{\mathcal{L}(\mathcal{X} + f)}(d) = h\mathcal{X}(d) - h\mathcal{X}(d - e) \]

\[ = \frac{d^m}{m!} \cdot \alpha + \frac{d^{m-1}}{(m-1)!} \cdot \beta + \ldots \]

\[ - \left( \frac{(d - e)^m}{m!} \cdot \alpha + \frac{(d - e)^{m-1}}{(m-1)!} \beta + \ldots \right) \]
Proof:

\[ h_{l(X)+(f)}(d) = h_X(d) - h_X(d - e) \]

\[ = \frac{d^m}{m!} \cdot \alpha + \frac{d^{m-1}}{(m-1)!} \cdot \beta + \ldots \]

\[ - \left( \frac{(d - e)^m}{m!} \cdot \alpha + \frac{(d - e)^{m-1}}{(m-1)!} \beta + \ldots \right) \]

Compute

\[ h(d) = \frac{\alpha}{m!} (d^m - (d - e)^m) + \frac{\beta}{(m-1)!} (d^{m-1} - (d - e)^{m-1}) + \ldots \]
Proof:

\[ h_{I(\chi)+(f)}(d) = h\chi(d) - h\chi(d - e) \]

\[ = \frac{d^m}{m!} \cdot \alpha + \frac{d^{m-1}}{(m-1)!} \cdot \beta + \ldots \]

\[ - \left( \frac{(d - e)^m}{m!} \cdot \alpha + \frac{(d - e)^{m-1}}{(m-1)!} \cdot \beta + \ldots \right) \]

Compute

\[ h(d) = \frac{\alpha}{m!}(d^m - (d - e)^m) + \frac{\beta}{(m-1)!}(d^{m-1} - (d - e)^{m-1}) + \ldots \]

\[ = \frac{\alpha}{(m-1)!} \cdot e \cdot d^{m-1} + \ldots \]
Proof:

\[ h_{I(X)+(f)}(d) = h_X(d) - h_X(d - e) \]

\[ = \frac{d^m}{m!} \cdot \alpha + \frac{d^{m-1}}{(m-1)!} \cdot \beta + \ldots \]

\[- \left( \frac{(d - e)^m}{m!} \cdot \alpha + \frac{(d - e)^{m-1}}{(m-1)!} \beta + \ldots \right) \]

Compute

\[ h(d) = \frac{\alpha}{m!} (d^m - (d - e)^m) + \frac{\beta}{(m-1)!} (d^{m-1} - (d - e)^{m-1}) + \ldots \]

\[ = \frac{\alpha}{(m-1)!} \cdot e \cdot d^{m-1} + \ldots \]

\[ \implies \deg I(X) + (f) = \alpha \cdot e = \deg X \cdot \deg f \]
Corollary

For $X$ irreducible curve, $f$ not vanishing identically on $X$:

$$\#X \cap Z(f) \leq \deg X \cdot \deg f$$
Corollary

For $X$ irreducible curve, $f$ not vanishing identically on $X$:

$$\# X \cap Z(f) \leq \deg X \cdot \deg f$$

Proof: $Z(I(X) + (f)) = X \cap Z(f)$
Corollary

For $X$ irreducible curve, $f$ not vanishing identically on $X$:

$$\# X \cap Z(f) \leq \deg X \cdot \deg f$$

Proof: $Z(I(X) + (f)) = X \cap Z(f)$

$$\# X \cap Z(f) \leq \deg I(X) + (f) = \deg X \cdot \deg f$$
Corollary

For $X$ irreducible curve, $f$ not vanishing identically on $X$:

$$\# X \cap Z(f) \leq \deg X \cdot \deg f$$

Proof: $Z(I(X) + (f)) = X \cap Z(f)$

$$\# X \cap Z(f) \leq \deg I(X) + (f) = \deg X \cdot \deg f$$

Corollary

If $X, Y$ are two curves in $\mathbb{P}^2$, without common components, degrees $d, e$:

$$\# X \cap Y \leq d \cdot e$$
Corollary

For $X$ irreducible curve, $f$ not vanishing identically on $X$:

$$\# X \cap Z(f) \leq \deg X \cdot \deg f$$

Proof: $Z(I(X) + (f)) = X \cap Z(f)$

$$\# X \cap Z(f) \leq \deg I(X) + (f) = \deg X \cdot \deg f$$

Corollary

If $X, Y$ are two curves in $\mathbb{P}^2$, without common components, degrees $d, e$:

$$\# X \cap Y \leq d \cdot e$$