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Figure: David Hilbert
Proof of strong Nullstellensatz

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- It is easy to see $\sqrt{i} \subset I_Z(i)$.
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Definition

A topological space \( X \) is reducible if \( X = X_1 \cup X_2 \) for two proper closed subsets \( X_1 \) and \( X_2 \).
Irreducibility

Remark: If \( X_1 \) and \( X_2 \) are required disjoint, \( X \) is said to be disconnected. A disconnected set it reducible.
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When $k = \mathbb{C}$, $\mathbb{A}^1$ is reducible in the usual topology. To see this, set

$$X_1 = \{z \in \mathbb{C} : |z| \geq 1\}, \ X_2 = \{z \in \mathbb{C} : |z| \leq 1\}.$$

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Definition
An irreducible affine algebraic set is called an affine variety.
Prime ideals

Affine algebraic sets are in 1 − 1 correspondence with radical ideals. How about affine varieties?
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An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible iff $I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$.
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Example What are the irreducible subsets of $\mathbb{A}^2$? What are the prime ideals of $k[X, Y]$?
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We claim

- $\mathfrak{p}$ is principal generated by one irreducible polynomial $f$
- $\mathfrak{p}$ is maximal, $\mathfrak{p} = (X - a, Y - b)$ for some $a, b \in k$. 

Geometrically, this means that the affine proper subvarieties of $\mathbb{A}^2$ are points and irreducible affine curves.
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Now,

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If $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map and $X$ is irreducible in $\mathbb{A}^n$, then $f(X)$ is also irreducible.
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- \[ X = X_1 \cup X_2 \implies X_1 = X \text{ or } X_2 = X. \]
  This means
  \[ X \subset f^{-1}(Y_i) \implies f(X) \subset Y_i \implies Z_i = X. \]
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Finiteness conditions

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A topological space $X$ is called Noetherian if every descending chain of closed subsets

$$X ⊃ X_1 ⊃ X_2 ⊃ \ldots$$

is stationary

Remark.
- Since $k[x_1, \ldots, x_n]$ is a Noetherian ring, then $A^n$ is a Noetherian topological space.
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Figure: Emmy Noether
Irreducible components

Theorem
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such that $X_i \not\subset X_j$ for $i \neq j$. The decomposition is unique up to reordering of the $X_i$’s.
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Analogy: any square-free integer is product of distinct primes.
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$X = \bigcup_i X_i = \bigcup_j Y_j$.

- $X_i = \bigcup_i (Y_j \cap X_i) \Rightarrow X_i \subset Y_j$ for some $j$.

- Similarly, $Y_j \subset X'_i$.

Thus $i = i'$ and $X_i = Y_j$. The sets $X_i$'s are a permutation of the $Y_j$'s.
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“Everyone knows what a curve is, until he has studied enough mathematics to become confused ...”
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- Terminology: curve, surface, threefold, etc.
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▶ This is false.

Let $X = \{a, b\}$. Give $X$ the topology whose closed sets are $\emptyset$, $\{a\}$, $X$. 

$X$ is irreducible of dimension 1, while $U = \{b\}$ is a dense open set of dimension 0.

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II. Functions and morphisms of algebraic sets
Coordinate rings

- We wish to define regular functions on affine varieties.
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  - holomorphic functions, differentiable functions etc.

\[ \mathcal{A}(X) = \mathbb{K}[x_1, \ldots, x_n]/ \mathcal{I}(X) = \text{integral domain} \]

- Any \( f \in \mathcal{A}(X) \) gives a polynomial function \( f: X \to \mathbb{K} \)
- This is independent of choices \( f_1, f_2 \in \mathcal{A}(X) \Rightarrow f_1 - f_2 \in \mathcal{I}(X) \Rightarrow f_1|_X = f_2|_X. \)
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Let $U \subset X$ be open. We define regular functions on $U$. 

Strategy: A regular function on $U$ must be regular at each $p \in U$. 

▶ Let $p \in X$. The local ring $O_{X,p} = \{ f, g : g(p) \neq 0, f, g \in A(X) \} \subset K(X)$. These are the regular functions at $p$. 

▶ Maximal ideal $m_{X,p} = \{ f : f(p) = 0 \} \subset A(X)$ 

$A(X)/m_{X,p} \cong k, f \mapsto f(p)$. 

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- Let $p \in X$. The local ring

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\mathcal{O}_{X,p} = \left\{ \frac{f}{g} : g(p) \neq 0, f, g \in A(X) \right\} \subset K(X).
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$$A(X)/m_{X,p} \cong k, \quad f \mapsto f(p)$$

- $\mathcal{O}_{X,p}$ is the localization

$$A(X)_{m_{X,p}}.$$
Regular functions on open subsets

- Regular functions on $U$:

$$\mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p} \subset K(X).$$

Beware: this is trickier than it looks!!!

Not all regular functions on $U$ can be expressed globally as quotients of two polynomials!!!!

Example: Let $X = \{ xw - yz = 0 \} \subset \mathbb{A}^4$ and $U = \{ yw \neq 0 \}$. The function $\phi = \begin{cases} x & \text{for } y \neq 0 \\ z & \text{for } w \neq 0 \end{cases}$ is well-defined and regular on $U$. It is not a global quotient of two polynomials.
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Regularity is local

Lemma

Let $U \subset X \subset \mathbb{A}^n$ be open. Let

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\[ \phi : U \rightarrow k \]

be a set theoretic map. TFAE

- \( \phi : U \rightarrow k \) is regular at the point \( P \in U \)

Remark: A well-defined function on \( U \) is regular if it can be written locally as a quotient. That is, there exists a cover \( U = \bigcup_i U_i \) with \( \phi = f_i / g_i \) over \( U_i \), \( g_i \) never vanishing on \( U_i \).
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- $\phi : U \rightarrow k$ is regular at the point $P \in U$
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Remark: A well-defined function on $U$ is regular if it can be written locally as a quotient.
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**Lemma**

Let $U \subset X \subset \mathbb{A}^n$ be open. Let

$$\phi : U \to k$$

be a *set theoretic* map. TFAE

- $\phi : U \to k$ is *regular* at the point $P \in U$
- there is a *neighborhood* $V$ of $P$ in $U$, *polynomials* $f, g$ with
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Proof:

- **Forward:** We have $\phi = \frac{f}{g}$ with $g(p) \neq 0$. Let

  $$V = \{q \in U : g(q) \neq 0\}.$$ 

  This is open in $U$ and contains $p$. 

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- **Converse:** to each \( \phi = \frac{f}{g} \) in \( V \), associate \( \frac{f}{g} \in K(X) \).
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- **Let** $\phi = \frac{f}{g}$, $\phi = \frac{f'}{g'}$ in $V$ and $V'$.
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- **Forward:** We have $\phi = \frac{f}{g}$ with $g(p) \neq 0$. Let
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- Let $\phi = \frac{f}{g}, \phi = \frac{f'}{g'}$ in $V$ and $V'$. Let $W = V \cap V'$.
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- **Forward:** We have $\phi = \frac{f}{g}$ with $g(p) \neq 0$. Let

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