Math 203A – October 24, 2018

- Office hours today only: 1:45 - 2:45, also after 4:45
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Winter Quarter schedule?
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Last time

- morphisms of projective varieties
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- \( X, Y \) projective \( \implies X \times Y \) is projective

Loose ends:
- \( X \) projective variety \( \implies X \) is a variety
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- $X$ projective variety $\implies X$ is a variety
Lemma

Any *irreducible* projective algebraic set is a *variety*.
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$$\Delta \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

is closed.
Lemma

Any \textit{irreducible} projective algebraic set is a \textit{variety}.

Proof: Suffices to show that \( \mathbb{P}^n \) is a \textit{variety}. That is, we show

\[
\Delta \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n
\]

is \textit{closed}. Note

\[
\Delta = \{ (x, y) : [x_0 : \ldots : x_n] = [y_0 : \ldots : y_n] \} = \{ x_i y_j - x_j y_i = 0 \}
\]
Lemma
Any irreducible projective algebraic set is a variety.

Proof: Suffices to show that $\mathbb{P}^n$ is a variety. That is, we show

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$$\Delta = \{(x, y) : [x_0 : \ldots : x_n] = [y_0 : \ldots : y_n]\} = \{x_iy_j - x_jy_i = 0\}$$

which is given by bihomogeneous equations.
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Intermezzo: Grassmannians

- We consider a cousin of projective space: the Grassmannian
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- $G(1, n)$ is the \textit{set of lines} in $\mathbb{P}^n$ or \textit{equivalently} 2-planes through the origin in $\mathbb{A}^{n+1}$
Intermezzo: Grassmannians

- We consider a cousin of projective space: the Grassmannian

- $G(1, n)$ is the set of lines in $\mathbb{P}^n$ or equivalently 2-planes through the origin in $\mathbb{A}^{n+1}$

Figure: Herman Grassman

Figure: Julius Plücker
Theorem

The Grassmannian \(G(1, n)\) is a projective variety of dimension \(2(n - 1)\).
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Strategy:

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- Projectivity:
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\[ \Phi : G(1, n) \hookrightarrow \mathbb{P}^N \]
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- The discussion applies to $G(k, n)$. 
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The Grassmannian $G(1, n)$ is a projective variety of dimension $2(n - 1)$.

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- $G(1, n)$ is prevariety - affine cover
- Projectivity: Plücker morphism
  \[ \Phi : G(1, n) \hookrightarrow \mathbb{P}^N \]
- The discussion applies to $G(k, n)$. Dimension $(k + 1)(n - k)$. 
Plucker morphism

Let $L = \mathbb{P}(V)$ with $V \subset \mathbb{A}^{n+1}$ be spanned by $a, b$ with

$$a = (a_0, \ldots, a_n), \quad b = (b_0, \ldots, b_n)$$
Plucker morphism

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\begin{align*}
a &= (a_0, \ldots, a_n), & b &= (b_0, \ldots, b_n)
\end{align*}
\]

Define

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\Phi : G(1, n) \to \mathbb{P}(\wedge^2 k^{n+1}), \quad L \mapsto [a \wedge b]
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a \wedge b = \left( \sum a_ie_i \right) \wedge \left( \sum b_je_j \right)
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These are the 2 \times 2 minors of the matrix with rows \( a \) and \( b \)
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This is independent of choices. If $a', b'$ is a new basis with change of matrix $A$.
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**Injectivity:** Let

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V = \langle a, b \rangle, \ V' = \langle a', b' \rangle.
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\[ \Phi(L) = \Phi(L') \implies \]

\[ a \wedge b = a' \wedge b' \implies a, b, a', b' \text{ linearly dependent} \]

Similarly, \( a, b, b' \) dependent.

Thus \( L = L' \).
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\Phi(L) = \Phi(L') \implies a \wedge b = a' \wedge b' \implies a \wedge b \wedge a' = 0
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\[
\implies
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Thus $L = L'$. 
$G(1, n)$ as a prevariety

- **Claim:** cover image of $\Phi$ by affine opens $\cong \mathbb{A}^{2(n-1)}$. 
$G(1, n)$ as a prevariety

- Claim: cover image of $\Phi$ by affine opens $\simeq \mathbb{A}^{2(n-1)}$.
- WLOG, the coordinate corresponding to $e_0 \wedge e_1$ in $\Phi(L)$ is $\neq 0$. 

The first $2 \times 2$ minor of the matrix

\[
\begin{pmatrix}
a_0 & \ldots & a_n \\
b_0 & \ldots & b_n
\end{pmatrix}
\]

is non-zero.

The Gaussian algorithm brings this matrix into the form

\[
\begin{pmatrix}
1 & 0 & a'_2 & \ldots & a'_n \\
0 & 1 & b'_2 & \ldots & b'_n
\end{pmatrix}
\]

The affine open $L \to (a'_2, \ldots, a'_n, b'_2, \ldots, b'_n) \in \mathbb{A}^{2(n-1)}$. 

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- **Claim:** cover image of $\Phi$ by affine opens $\sim \mathbb{A}^{2(n-1)}$.
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G(1, n) as a prevariety

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Projectivity

- We use an algebraic fact:

  \[ \omega \in \Lambda^2 V \text{ splits as } \omega = a \wedge b \iff \omega \wedge \omega = 0. \]

- Let

  \[ \omega = \sum \omega_{ij} e_i \wedge e_j. \]
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= \sum_{i<j<k<l} \left( \omega_{ij} \omega_{kl} - \omega_{ik} \omega_{jl} + \omega_{il} \omega_{jk} \right) e_i \wedge e_j \wedge e_k \wedge e_l
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- The image of \( \Phi \) is cut by the quadrics

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- \( G(1, n) \) is projective
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- The image of \( \Phi \) is cut by the quadrics
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- \( G(1, n) \) is projective
Example:

\[ G(1, 3) \] is the set of lines in \( P^3 \).

\[ G(1, 3) \] is the quadric in \( P^5 \):

\[ \omega_{12} \omega_{34} - \omega_{13} \omega_{24} + \omega_{14} \omega_{23} = 0. \]

The line \( 5x + 2y = 3z + w = 0 \) in \( P^3 \).

Let

\[ a = 2e_1 - 5e_2, \]
\[ b = e_3 - 3e_4. \]

Plücker coordinates

\[ a \wedge b = 2e_13 - 5e_23 - 6e_14 + 15e_24. \]
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Main theorem of projective varieties

Theorem

Let $X$ be projective, $Y$ any variety, and $f : X \to Y$ a morphism.

Remark:

$\text{▶}$ fails for affine $f : \mathbb{A}^2 \to \mathbb{A}^1$, $X = \{xy - 1 = 0\}$, $f(X)$ not closed.

$\text{▶}$ analogy: projective $\mapsto$ compact variety $\mapsto$ Hausdorff morphism $\mapsto$ continuous.
Main theorem of projective varieties

Theorem
Let $X$ be projective, $Y$ any variety, and $f : X \to Y$ a morphism. Then

$$f(X) \text{ is closed in } Y.$$
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If $X, Y$ are projective, then $f(X)$ is projective.
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- analogy:
  $\text{projective} \leftrightarrow \text{compact}$
  $\text{variety} \leftrightarrow \text{Hausdorff}$
  $\text{morphism} \leftrightarrow \text{continuous}$
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$$f : \mathbb{A}^2 \to \mathbb{A}^1, \ X = \{xy - 1 = 0\}, \ f(X) \text{ not closed.}$$

▷ analogy:

projective $\mapsto$ compact

variety $\mapsto$ Hausdorff

morphism $\mapsto$ continuous
Corollary

If $X$ is irreducible, projective and $f : X \to k$ is regular, then $f$ is constant.
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Basic open sets

Lemma

Let $X \subset \mathbb{P}^n$ be projective variety.

The basic open set $X_f = \{ x \in X : f(x) \neq 0 \}$ is an abstract affine variety.

Corollary

In particular, unless $X$ is a point, $X \cap Z(f) \neq \emptyset$, so $X$ intersects any hypersurface.

If $X \cap Z(f) = \emptyset$ then $X_f = X$ would be affine.

Affine varieties have lots of regular functions. Projective varieties only have constants.

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  Since $\nu(X)$ projective and $\ell$ linear, $\nu(X_f)$ is affine, so $X_f$ is affine.
Strategy of proof

$X$ projective, $Y$ variety, $f : X \to Y$ morphism $\implies f(X)$ closed.
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\[ X \text{ projective, } Y \text{ variety, } f : X \rightarrow Y \text{ morphism } \implies f(X) \text{ closed.} \]

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▸ in AG: complete/proper varieties

▸ show projective varieties $X$ are complete

▸ show any $f : X \to Y$ is a closed map, for $X$ complete, so also for $X$ projective