Math 203A
Last time

- **Strategy:** compute dimension by comparing with varieties whose dimension we already know
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- if $f : X \to Y$ surjective, finite fibers then

  $$\dim X = \dim Y.$$
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- if $f : X \to Y$ surjective, finite fibers then
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  \dim X = \dim Y.
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Projections

$\mathbb{P}^n, \quad X \subset \mathbb{P}^n, \quad X \neq \mathbb{P}^n, \quad p \in \mathbb{P}^n \setminus X, \quad p \notin H \subset \mathbb{P}^n$, hyperplane

$\pi: X \to H, \quad \pi(q) = \text{line } pq \cap H$, surjective onto its image, finite fibers

If $p = [0 : \ldots : 0 : 1]$, $H = \{x_n = 0\}$:

$\pi: \mathbb{P}^n \to \mathbb{P}^{n-1}, \quad \pi(q) = [q_0 : q_1 : \ldots : q_n - 1]$
Projections

- $X \subset \mathbb{P}^n$, $X \neq \mathbb{P}^n$
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- \( X \subset \mathbb{P}^n, X \neq \mathbb{P}^n, p \in \mathbb{P}^n \setminus X, \)
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Strategy:

- Prove

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\dim X = \dim \pi(X), \quad X \subset \mathbb{P}^n, \quad \pi(X) \subset \mathbb{P}^{n-1}
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- Continue.
Strategy:

- **Prove**

  \[ \dim X = \dim \pi(X), \quad X \subset \mathbb{P}^n, \quad \pi(X) \subset \mathbb{P}^{n-1} \]

- **Continue.** Eventually, the composition of projections

  \[ \pi : X \to \mathbb{P}^m \text{ surjective} \implies \dim X = \dim \mathbb{P}^m \]
Lemma

Let $X$ be a variety of dimension $n$.

(i) if $Y \varsubsetneq X$ is closed,

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is the longest chain, then $\dim X_i = i$
First properties

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(i) If $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_m = Y$ is the longest chain in $Y$, then
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If \( \dim X_i > i \), take the longest chain in \( X_i \).

Adjoin \( X_{i+1}, \ldots, X_n \) to get a chain in \( X \).
(ii) By (i), $\dim X_i \geq i$.

If $\dim X_i > i$, take the longest chain in $X_i$.

Adjoin $X_{i+1}, \ldots, X_n$ to get a chain in $X$ of length

$$\dim X_i + (n - i) > n.$$
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Contradiction!
Surjective maps

Lemma
Let $f : X \to Y$ be a surjective morphism of projective varieties.
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Every longest chain in $Y$:

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = Y$$
Lemma

Let $f : X \to Y$ be a surjective morphism of projective varieties.

Every longest chain in $Y$: $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = Y$

can be lifted to a chain in $X$: $X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$

with $f(X_i) = Y_i$. 

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Lemma

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with \( f(X_i) = Y_i \). Thus

\[
\dim X \geq \dim Y.
\]
Proof:

- **Induct** on dim $Y$. 

---

Proof (continued):

- **Inductive step**: $\dim Y = n - 1 = n - 1$.

- Irreducible components $Z = f^{-1}(Y_{n-1}) = Z_1 \cup \ldots \cup Z_r$.

- $Y_{n-1} = f(Z_1) \cup \ldots \cup f(Z_r)$.

- $Z$ is closed in $X$, $Z_j$ closed in $Z$, thus $Z_j$ projective.

- $f(Z_j)$ is irreducible, projective, $Y_{n-1}$ is irreducible.

- Apply induction to $f(Z_j) \rightarrow Y_{n-1}$.

- Lift $Y_0, \ldots, Y_{n-1}$ to chain in $Z_j$.

- Complete it by adjoining $X$. 

Proof:

- **Induct** on $\dim Y$.

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Proof:

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- \( Y_{n-1} = f(f^{-1}(Y_{n-1})) = f(Z_1) \cup \ldots \cup f(Z_r) \)
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- \( Y_{n-1} = f(f^{-1}(Y_{n-1})) = f(Z_1) \cup \ldots \cup f(Z_r) \)
- \( Z \) is closed in \( X \).
Proof:

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- $Y_{n-1} = f(f^{-1}(Y_{n-1})) = f(Z_1) \cup \ldots \cup f(Z_r)$
- $Z$ is **closed** in $X$, $Z_j$ **closed** in $Z$, thus $Z_j$ **projective**
Proof:

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- $f(Z_j)$ is irreducible,
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\[ f(Z_j) = Y_{n-1} \text{ for some } j \]
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Discussion

Let $X$ be projective. We showed

$$\dim \pi(X) \leq \dim X.$$
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If distinct, then we’re OK: $\dim X \leq \dim \pi(X)$. 
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Noether normalization

Problem: Given $Y \subset X$ closed, show $\pi(Y) \neq \pi(X)$. 

*Proposition (Noether normalization*)

Let $X \subset \mathbb{P}^n$ be irreducible, and $p \not\in X$, $p = [0 : \ldots : 0 : 1]$.

Let $f \in k[x_0, \ldots, x_n]$ homogeneous.

There exists $D > 0$ and $a_1, \ldots, a_D \in k[x_0, \ldots, x_n-1]$ homogeneous such that $f^D + a_1 f^{D-1} + \ldots + a_D = 0$ on $X$. 

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Solution: Find a polynomial vanishing on $\pi(Y)$, but not on $\pi(X)$.
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such that

$$f^D + a_1 f^{D-1} + \ldots + a_D = 0 \text{ on } X.$$
Expanded solution:

\[ f \in I(Y) \setminus I(X) \]

\[ f(D^+ a_1 f(D^- 1 + ...) + a(D^+ \pi(y)) f(D^- 1 + ...) + a(D^\pi(Y)) = 0 \] on \( X \) with \( D \) minimal.

\[ f(y) D^+ a_1 (\pi(y)) f(D^- 1 + ...) + a(D^\pi(Y)) = 0 \] since \( f(y) = 0 \) for \( y \in Y \), \( a(D^\pi(Y)) = 0 \) \( \Rightarrow a(D^\pi(Y)) = 0 \) on \( \pi(Y) \).

\[ f(D^+ a_1 f(D^- 1 + ...) + a(D^\pi(Y))) = 0 \] on \( X \).

\( S(X) \) is integral domain, and \( f \neq 0 \) in \( S(X) \).
Expanded solution: Pick $f$ homogeneous, $f \in \mathcal{I}(Y) \setminus \mathcal{I}(X)$. 
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  with $D$ minimal.

- For $y \in Y \subset X$
  \[ f(y)^D + a_1(\pi(y)) \cdot f(y)^{D-1} + \ldots + a_D(\pi(y)) = 0 \]
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- Since \( f(y) = 0 \) for \( y \in Y \),
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  \[ a_D(\pi(y)) = 0 \implies a_D = 0 \text{ on } \pi(Y). \]
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  \[ a_D(\pi(y)) = 0 \implies a_D = 0 \text{ on } \pi(Y). \]

- If $a_D = 0$ on $\pi(X)$ then
  \[ f^D + a_1 f^{D-1} + \ldots + a_{D-1} f = 0 \text{ on } X. \]
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- Since $f(y) = 0$ for $y \in Y$,
  
  $$a_D(\pi(y)) = 0 \implies a_D = 0 \text{ on } \pi(Y).$$

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  with $D$ minimal.

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  f(y)^D + a_1(\pi(y)) \cdot f(y)^{D-1} + \ldots + a_D(\pi(y)) = 0
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- Since $f(y) = 0$ for $y \in Y$,
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- If $a_D = 0$ on $\pi(X)$ then
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- $S(X)$ is integral domain, and $f \neq 0$ in $S(X)$. 
Expanded solution: Pick $f$ homogeneous, $f \in I(Y) \setminus I(X)$.

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Let $X \subset \mathbb{P}^n$ be irreducible, and $p \not\in X$, $p = [0 : \ldots : 0 : 1]$. 

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▶ The fibers of $\pi : X \to \mathbb{P}^n - 1$ have $\leq D$ points.

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