Math 203A

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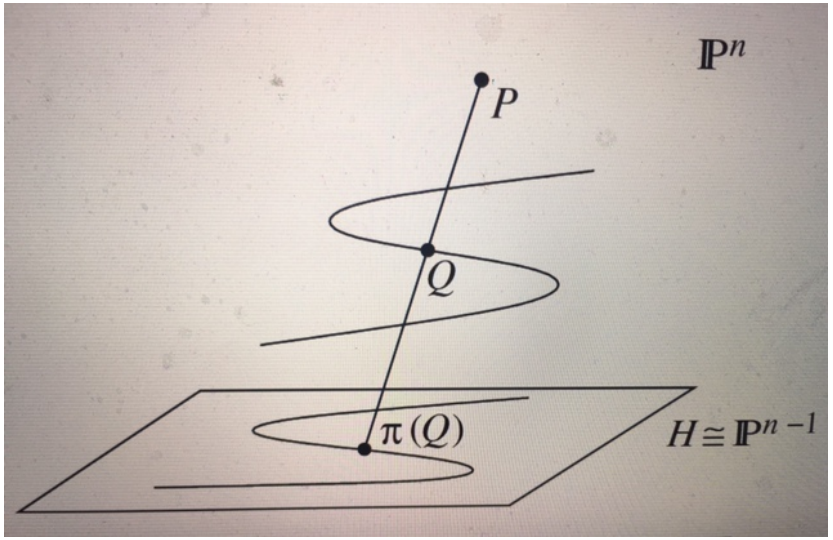

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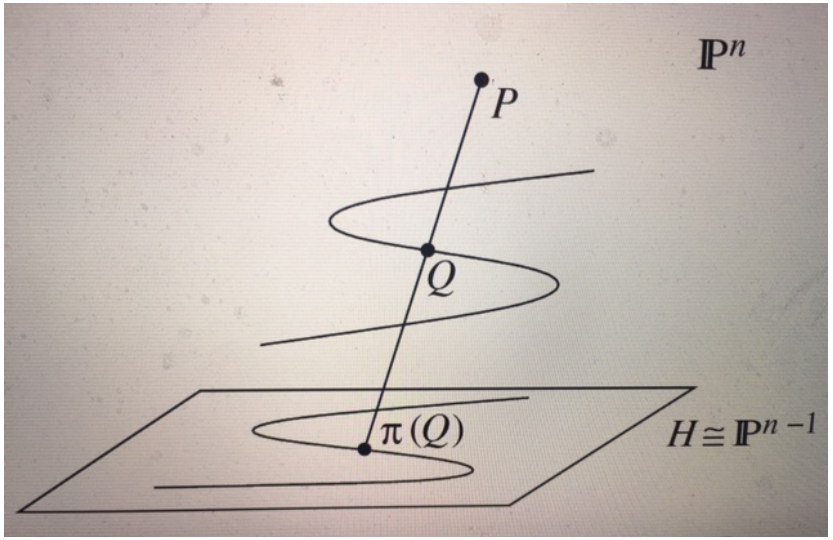

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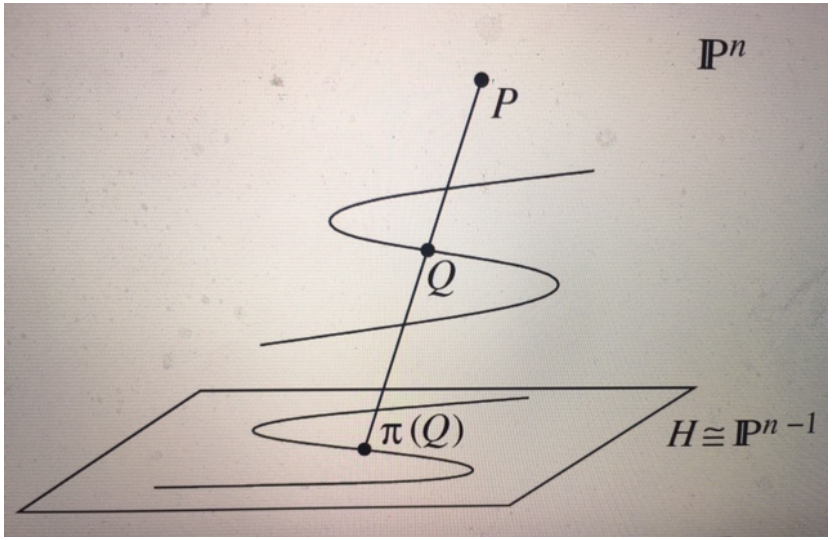

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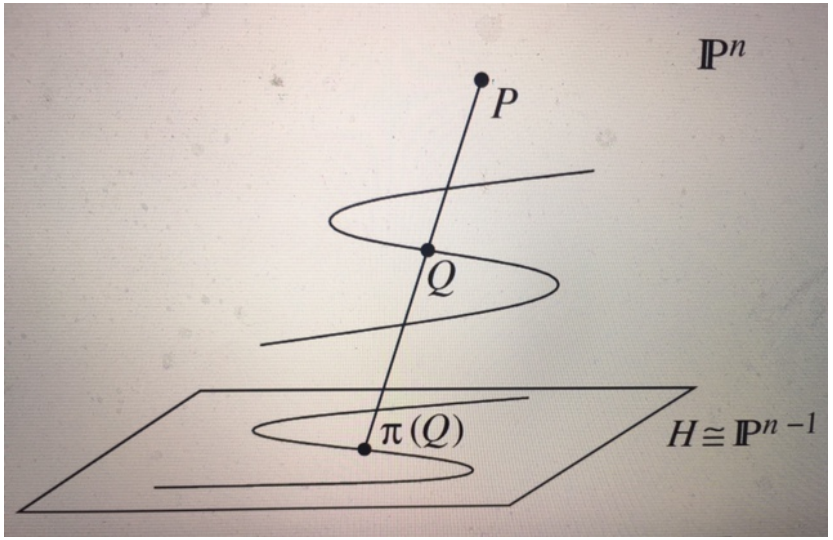

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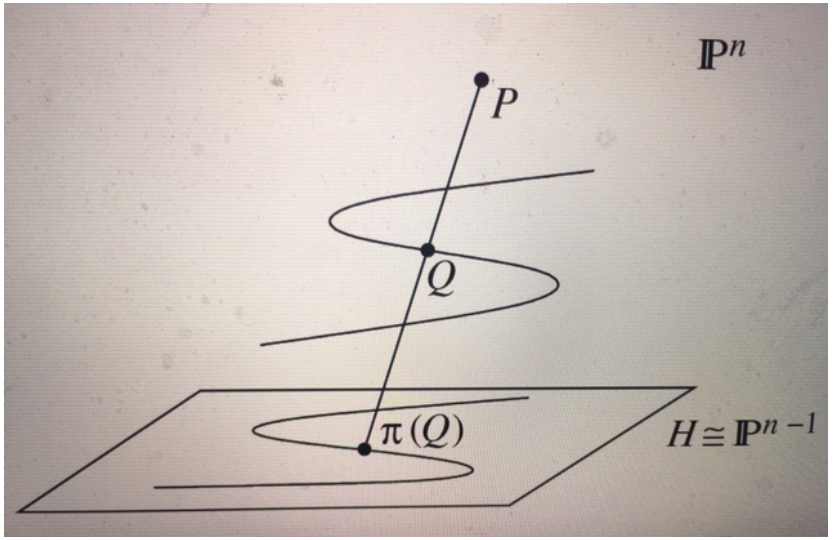

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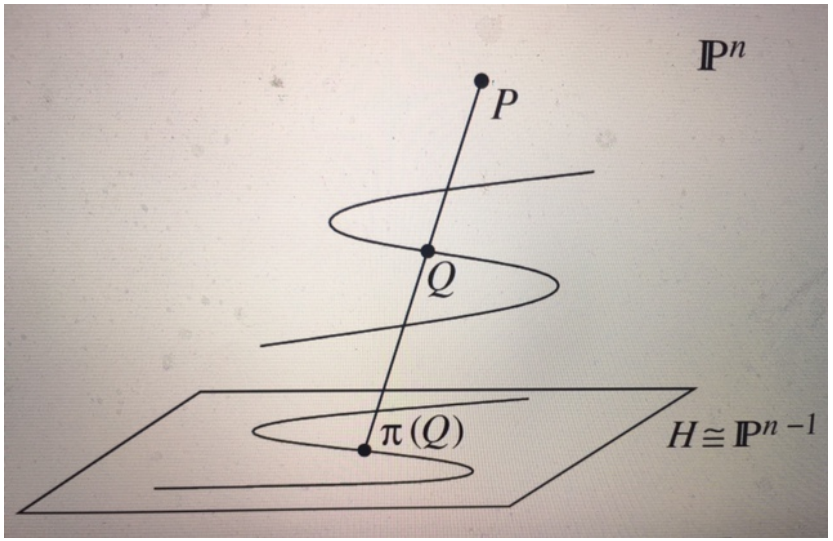

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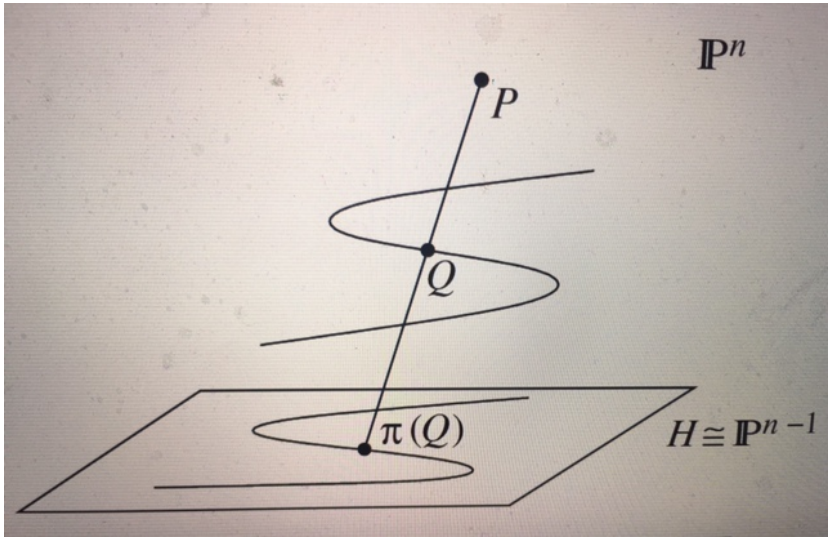

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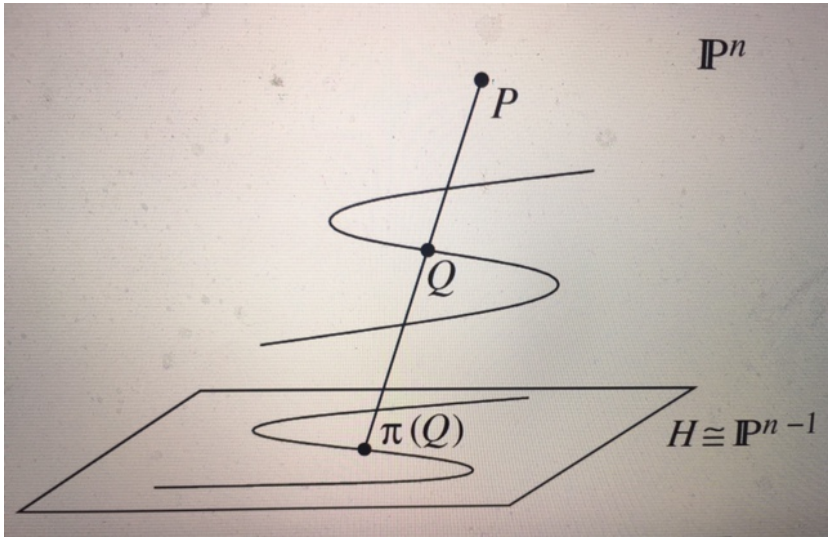

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- Continue. Eventually, the composition of projections

$$\pi : X \to \mathbb{P}^m \text{ surjective } \implies \dim X = \dim \mathbb{P}^m$$

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Let X be a variety of dimension n.

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Contradiction!

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Let $f \in k[x_0, \ldots, x_n]$ homogeneous. There exists $D > 0$ and

$$a_1, \ldots, a_D \in k[x_0, \ldots, x_{n-1}] \text{ homogeneous}$$

such that

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$$f^D + a_1 f^{D-1} + \ldots + a_D = 0 \text{ on } X$$
with D minimal.
- For $y \in Y \subset X$
$$f(y)^D + a_1(\pi(y)) \cdot f(y)^{D-1} + \ldots + a_D(\pi(y)) = 0$$
- Since $f(y) = 0$ for $y \in Y$,
$$a_D(\pi(y)) = 0 \implies a_D = 0 \text{ on } \pi(Y).$$
- If $a_D = 0$ on $\pi(X)$ then
$$f^D + a_1 f^{D-1} + \ldots + a_{D-1} f = 0 \text{ on } X.$$
- $S(X)$ is integral domain, and $f \neq 0$ in $S(X)$.

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Next: proof and discussion of Noether normalization.

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