Math 203A
Proposition (Noether normalization)

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▶ Geometrically, $\pi: X \to \pi(X)$ is quasi-finite.

▶ We would like to explore this further.
Last time

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Let $f \in k[x_0, \ldots, x_n]$ homogeneous.
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$$a_1, \ldots, a_D \in k[x_0, \ldots, x_{n-1}]$$

homogeneous such that

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on $X$. 

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$q$ maximal $\iff p$ maximal.
**Geometrically:** Let $f : X \rightarrow Y$ be dominant morphism of affine varieties, and

$$f^* : A(Y) \rightarrow A(X).$$
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Algebra is the offer made by the devil to the mathematician.

The devil says: ‘I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.’

M. Atiyah
Intuitively: If $X \subset \mathbb{A}^n$, the coordinate function $t_i \in A(X)$ satisfies

$$t_i^k + a_1 \cdot t_i^{k-1} + \ldots + a_k = 0$$

for $a_i \in A(Y)$. Thus $t_i(x)$ takes on finitely many values.

As $y$ varies, the points in $f^{-1}(y)$ can come together but cannot disappear.
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Lemma

For a finite morphism of affine varieties $f : X \to Y$

Proof:

(i) $A(Y) \hookrightarrow A(X)$ integral $\Rightarrow$ satisfies going up

If $y \in Y$, let $m_y$ be the maximal ideal in $A(Y)$.

Let $n$ maximal ideal in $A(X)$ such that $n \cap A(Y) = m_y$.

$n$ corresponds to $x \in X$.

Check: $f(x) = y \iff m_x \cap A(Y) = m_y$.

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- Thus \( f(x) = y \).
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Finite maps in general

If $X, Y$ are varieties, $f : X \rightarrow Y$ is finite
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If $X$, $Y$ are varieties, $f : X \to Y$ is finite if there exists an affine cover $V_\alpha$ of $Y$ s.t.
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Rephrasing of Noether normalization

**Theorem**

Let \( p \notin X \subset \mathbb{P}^n \). The *projection* away from \( p \)

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\pi : X \to \pi(X)
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is a *finite morphism*. 
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W = \{ x_0 = 1 \} \simeq \mathbb{A}^n \subset \mathbb{P}^n, \quad U = W \cap X \text{ affine open in } X.
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V = \pi(U) = \pi(X) \cap \mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^{n-1} \text{ is closed, hence affine}.
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We show \( \pi : U \to V \) is a **finite map**.
If $f(x_1, \ldots, x_n)$ regular on $U$, form $\tilde{f}(x_0, \ldots, x_n)$ homogeneous.
If \( f(x_1, \ldots, x_n) \) regular on \( U \), form \( \bar{f}(x_0, \ldots, x_n) \) homogeneous.

By lemma,

\[
\bar{f}^D + \bar{a}_1 \bar{f}^{D-1} + \ldots + \bar{a}_D = 0 \text{ on } X.
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By lemma,

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$$x_0 = 1, \bar{a}_i = a_i(1, x_1, \ldots, x_n)$$

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Thus all \( f \in A(U) \) satisfy a monic equation with coefficients in \( a_i \in A(V) \).
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Noether normalization

- If $X$ is projective, there exists a finite morphism
  \[ \pi : X \to \mathbb{P}^m, \]
  where $m = \text{dim } X$.

- If $X$ is affine, there exists a finite morphism
  \[ \pi : X \to \mathbb{A}^m. \]
Roadmap

We know \( \dim \mathbb{P}^n = n \).
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Goals:
- Show \( \dim X \cap Z(f) = \dim X - 1 \).
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Goals:
- Show \( \dim X \cap Z(f) = \dim X - 1 \).
- Discuss affine varieties and arbitrary varieties.
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There could potentially be components of smaller dimension.

Reduction to linear \( f \): If \( \deg f = d \), use Veronese \( v : \mathbb{P}^n \rightarrow \mathbb{P}^{N} \).

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Dimension drops by 1 on RHS, \( v \) is isomorphism, so same is true on LHS.
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Let $X \subset \mathbb{P}^n$ projective variety, and $f \in k[x_0, \ldots, x_n]$ non-constant homogeneous, not identically zero on $X$.

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Why possible? First condition is clear, second condition automatic by looking at $X_{i+1}$. 

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