Math 203A

Last time

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Algebra is the offer made by the devil to the mathematician.

The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'

M. Atiyah

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Noether normalization

- If X is projective, there exists a finite morphism

$$\pi : X \to \mathbb{P}^m,$$

  where $m = \dim X$.
- If X is affine, there exists a finite morphism

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Remark: This means one irreducible component of $X \cap Z(f)$ has dimension $\dim X - 1$.

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