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Dimension of arbitrary varieties

- we pass from projective varieties to affine varieties.
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**Theorem**

Let $U \subset X$ be nonempty open, $X$ variety. Then

$$\dim U = \dim X.$$
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*Let $U \subseteq X$ be nonempty open, $X$ variety. Then*

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**Remarks:**

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This holds for all components.
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Let $X$ be a variety. Then

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- $X$ compact complex manifold/space

Question: $\dim X = \text{tr. deg. } \mathbb{C}^M(X)$?

Answer: $X$ is Moishezon manifold

- complex algebraic varieties are Moishezon
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- WTS: \( \dim U \geq \dim X \):
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  - Construct \( U_j = X_j \cap U \), \( U_0 \subset U_1 \subset \ldots \subset U_n = U \).

Issues: repetitions, \( U_0 \) may be empty.
Proof of the Theorem: $\dim U = \dim X$.

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(i) The case $X_0 \subset U$
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Step (i)

If $X_0 \subset U$ we show $U_j = X_j \cap U$ is a chain in $U$. 
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This is false since $X_0$ is contained in LHS but not RHS.
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By irreducibility,

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X_{j+1} = X_{j+1} \cap Z \implies X_{j+1} \subset Z.
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Step (ii)

Let $X$ projective.

Construct longest chain $\emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n = X$ by downward induction.

Let $X_n = X$.

Given $X_n, \ldots, X_j$ with $\dim X_j = j$ and $X_j \cap U \neq \emptyset$ construct $X_j-1$ with $\dim X_j-1 = j-1$ and $X_j-1 \cap U \neq \emptyset$.

Pick a non-constant homogeneous polynomial $f$ that is not zero on any irreducible component of $X_j \cap Z$.

A component of $X_j \cap Z (f)$ has dimension $\dim X_j-1 = j-1$. Call it $X_j-1$. 

\[ \text{Step (ii)} \]
\[ X \text{ projective.} \]
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$X$ projective. Construct longest chain

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\( X_{j-1} \) with \( \dim X_{j-1} = j - 1 \) and \( X_{j-1} \cap U \neq \emptyset \).

- Pick a non-constant homogeneous polynomial \( f \) that is not zero on any irreducible component of \( X_j \cap Z \).
- A component of \( X_j \cap Z(f) \) has dimension \( \dim X_j - 1 = j - 1 \).
Step (ii)

$X$ projective. Construct longest chain

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$$

with $X_0 \subset U$ by downward induction.

Let $X_n = X$. Given

$$X_n, \ldots, X_j$$ with $\text{dim} \ X_j = j$ and $X_j \cap U \neq \emptyset$

construct

$$X_{j-1}$$ with $\text{dim} \ X_{j-1} = j - 1$ and $X_{j-1} \cap U \neq \emptyset$.

- Pick a non-constant homogeneous polynomial $f$ that is not zero on any irreducible component of $X_j \cap Z$.
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Intersections with hypersurfaces - Part II

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Algebraic preliminaries: Take

- $B \hookrightarrow A$ extension of integral domains

Each $a \in B$ gives $L$-linear map $\mu_a: K \to K$, $k \mapsto ak$. Set $N_{K/L}(a) = \det(\mu_a) \in L$. 
Algebraic preliminaries: Take

- $B \hookrightarrow A$ extension of integral domains
- field extension

$L \hookrightarrow K$ is finite.

Define $N_{K/L}: K \ni \star \mapsto L \ni \star$. Each $a \in K$ gives $L$-linear map $\mu_a: K \rightarrow K$, $k \mapsto a \cdot k$.

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Consider $\pi : X \to \mathbb{A}^m$: 

$B = A(A_m), A = A(X) \to B \hookrightarrow A$.

Let $g = N(f) \in B \Rightarrow g$ regular on $A_m$.

Claim: $\mathbb{Z}(g) \subset \pi(X \cap \mathbb{Z}(f))$.

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