Math 203A
Recap:

- we proved $\dim X = \text{tr. deg.} K(X)$
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- intersections with hypersurfaces: projective or affine case
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Still to be done:

- loose ends
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- theorem of dimension of fibers
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- loose ends
- theorem of dimension of fibers
Last time

Theorem

$X$ affine variety, $f \in k[x_1, \ldots, x_n]$ polynomial that does not vanish identically on $X$. 
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\( X \) affine variety, \( f \in k[x_1, \ldots, x_n] \) polynomial that does not vanish identically on \( X \). Then

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\dim X \cap Z(f) = \dim X - 1
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if nonempty.
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When nonempty, all components of \( X \cap Z(f) \) have dimension \( \dim X - 1 \).
Corollary

An affine variety, \( f \in k[x_1, \ldots, x_n] \) polynomial that does not vanish identically on \( X \).

When nonempty, all components of \( X \cap Z(f) \) have dimension \( \dim X - 1 \).

Proof: Let

\[ X \cap Z(f) = Z_1 \cup \ldots \cup Z_r. \]
Corollary

$X$ affine variety, $f \in k[x_1, \ldots, x_n]$ polynomial that does not vanish identically on $X$.

When nonempty, all components of $X \cap Z(f)$ have dimension $\dim X - 1$.

Proof: Let

$$X \cap Z(f) = Z_1 \cup \ldots \cup Z_r.$$ 

WTS: $\dim Z_1 = \dim X - 1$. 
Corollary

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Proof: Let

\[ X \cap Z(f) = Z_1 \cup \ldots \cup Z_r. \]

WTS: \( \dim Z_1 = \dim X - 1 \). Let \( g \in I(Z_2 \cup \ldots \cup Z_r) \setminus I(Z_1) \).
Corollary

*X* affine variety, *f* ∈ *k*[*x*₁,..., *x*ₙ] polynomial that does not vanish identically on *X*.

When nonempty, all components of *X* ∩ *Z*(*f*) have dimension *dim* *X* − 1.

Proof: Let

\[ *X* \cap *Z*(*f*) = *Z*_1 \cup \ldots \cup *Z*_r. \]

WTS: *dim* *Z*_1 = *dim* *X* − 1. Let *g* ∈ *I*(*Z*_2 \cup \ldots \cup *Z*_r) \setminus *I*(*Z*_1).

\[ U = *X*_g \text{ is affine.} \]

\[ U \cap *Z*(*f*) = U \cap *Z*_1 \]
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X is affine variety. When nonempty,

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for all components.
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holds for all varieties X and \( f_i \) regular functions
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- holds for projective $X$ and $f_i$ homogeneous
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\( X, Y \subset \mathbb{A}^n \) affine varieties. When nonempty,

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\dim X \cap Y \geq \dim X + \dim Y - n
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$$X \cap Y = (X \times Y) \cap \Delta, \quad \Delta = Z(x_1 - y_1, \ldots, x_n - y_n)$$
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also true for projective varieties (hwk)
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- also true for projective varieties (hwk)
Theorem of dimension of fibers

**Theorem**

Let $f : X \to Y$ be a surjective morphism between varieties with

$$\dim X = n, \quad \dim Y = m.$$
Theorem of dimension of fibers

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Example: for equidimensional fibers

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Example: The incidence variety

\[ J = \{(p, L) : p \in L \subset \mathbb{P}^n\} \subset \mathbb{P}^n \times \mathbb{G}(1, n). \]
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Let $y \in Y$. We show $\dim F \geq n - m$ for all $F \hookrightarrow f^{-1}(y)$. 
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Claim:

- there is $U \subset Y$ affine,
- $f_1, \ldots, f_m$ regular with
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\textbf{Claim:}

\begin{itemize}
  \item there is $U \subset Y$ affine,
  \item $f_1, \ldots, f_m$ regular with
  \[
  U \cap Z(f_1, \ldots, f_m) = \{y\}
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\end{itemize}
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Proof of (i):

\[ f^{-1}(y) = f^{-1}(U) \cap Z(f^*f_1, \ldots, f^*f_m) \]
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- Arrange $Y \cap Z(f_1, \ldots, f_m)$ finite.
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$$Y^{(1)} = Y \cap Z(f)$$

which has components of dim $\leq \dim Y - 1$. 
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▸ If $Y \cap Z(f_1, \ldots, f_m) = \{y, y_1, \ldots, y_p\}$
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Let

$$U = Y_g = \text{affine}, \ U \cap Z(f_1, \ldots, f_m) = \{y\}.$$
Proof of (ii): WLOG $Y$ affine. Let $V \subset X$ dense affine open.
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tr. deg. $K(V)/K(Y)$
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- \( v_1, \ldots, v_N \) coordinates on \( V \subset \mathbb{A}^N \)
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- $v_1, \ldots, v_N$ coordinates on $V \subset \mathbb{A}^N$
- $y_1, \ldots, y_M$ coordinates on $Y \subset \mathbb{A}^M$
- WLOG: $v_1, v_2, \ldots, v_e$ are algebraically independent over $K(Y)$
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- $v_{e+1}, \ldots, v_n$ algebraically dependent on $K(Y)[v_1, \ldots, v_e]$. 

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for $i > e$, find equations

$$F_i(v_i; v_1, \ldots, v_e, y_1, \ldots, y_m) = 0$$
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$F_i$ are polynomials in $v_1, \ldots, v_e, v_i$ with coefficients functions in $y$
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F_i(v_i; v_1, \ldots, v_e, y_1, \ldots, y_m) &= 0
\end{align*}

\begin{itemize}
\item $F_i$ are polynomials in $v_1, \ldots, v_e, v_i$ with coefficients functions in $y$
\item Let $Y_i \subset Y$ be the vanishing of the leading term in $F_i$
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Let $Y_i \subset Y$ be the vanishing of the leading term in $F_i$

$$U = Y \setminus \bigcup Y_i \text{ open, nonempty}$$
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$F_i(T_i, T_1, \ldots, T_e, y)$ is non-zero polynomial for each fixed $y \in U$
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$F_i(T_i, T_1, \ldots, T_e, y)$ is non-zero polynomial for each fixed $y \in U$
Fix $y \in U$. Restrict $\overline{v}_i = v_i|_{f^{-1}(y) \cap V}$. 

\[ F_i(v_i, v_1, \ldots, v_e, y) = 0 \Rightarrow F_i(\overline{v}_i, \overline{v}_1, \ldots, \overline{v}_e) = 0 \quad \text{for} \quad i > e, \quad \overline{v}_i \text{ is algebraically dependent on } \overline{v}_1, \ldots, \overline{v}_e. \]

\[ \text{tr. deg.} K(f^{-1}(y)) \leq e \Rightarrow \dim f^{-1}(y) \leq e \Rightarrow \dim f^{-1}(y) = e. \]
Fix $y \in U$. Restrict $\overline{v}_i = v_i \big|_{f^{-1}(y) \cap V}$.

Restrict

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for $i > e$, $\bar{v}_i$ is algebraically dependent on $\bar{v}_1, \ldots, \bar{v}_e$
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tr. deg. $K(f^{-1}(y)) \leq e$
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Restrict

$$F_i(v_i; v_1, \ldots, v_e, y) = 0 \text{ to } f^{-1}(y) \cap V$$

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V. Smoothness
Goals:

- define the Zariski tangent space $T_pX$ and tangent cone $C_pX$
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- further topics: applications of smoothness, normal, factorial
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Zariski tangent space - affine case

Let $X \subset \mathbb{A}^n$, $p \in X$, 

\[ I = I(X) \subset k[x_1, \ldots, x_n]. \]

Any $f \in k[x_1, \ldots, x_n]$ satisfies:

\[ f = f(0) + f(1) + f(2) + \ldots. \]

$I(1) = \{ f(1) : f \in I \}$ is a vector subspace of $k[x_1, \ldots, x_n]$. 

The Zariski tangent space $T_p X = Z(I(1)) \subset \mathbb{A}^n$. 

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WLOG $p = (0, \ldots, 0)$,
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The Zariski tangent space

$$T_pX = Z(I^{(1)}) \subset \mathbb{A}^n$$
How to think about the tangent space

(i) The pairing

\[ k[x_1, \ldots, x_n]/I^{(1)} \times Z(I^{(1)}) \rightarrow k \]
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Intrinsic nature of Zariski tangent space

Lemma

Let $X \subset \mathbb{A}^n$, $p \in X$. Let

$$\mathfrak{m} \subset \mathcal{O}_{X,p} \text{ maximal ideal}, \; \mathfrak{m} = \{\phi : \phi(p) = 0\}.$$
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Remarks:

\( T_pX \) is independent of choice of coordinates
Intrinsic nature of Zariski tangent space

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Remarks:

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- for any variety $X$, $T_pX = T_pU$ for $U$ affine open near $p$. 

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