Math 203A

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- Fix $y \in U$. Restrict $\overline{v_i} = v_i\big|_{f^{-1}(y)\cap V}$.
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V. Smoothness

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(i) The pairing

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