Math 203A

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- functoriality

$$df \circ dg = d(g \circ f)$$

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Example: Projective space:

$$\mathbb{P}^n = \mathbb{P}(W), W = k^{n+1}, \quad L \subset W, \quad \dim L = 1$$

$$T_{[L]} \mathbb{P}^n = Hom(L, W/L)$$

Proof: Let $L = \langle v \rangle$. Given $f : L \to W/L$ construct

$$F : \mathbb{A}^1 \to \mathbb{P}^n, \quad t \mapsto L_t = \langle v + tf(v) \rangle.$$

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Figure: Eugenio Bertini

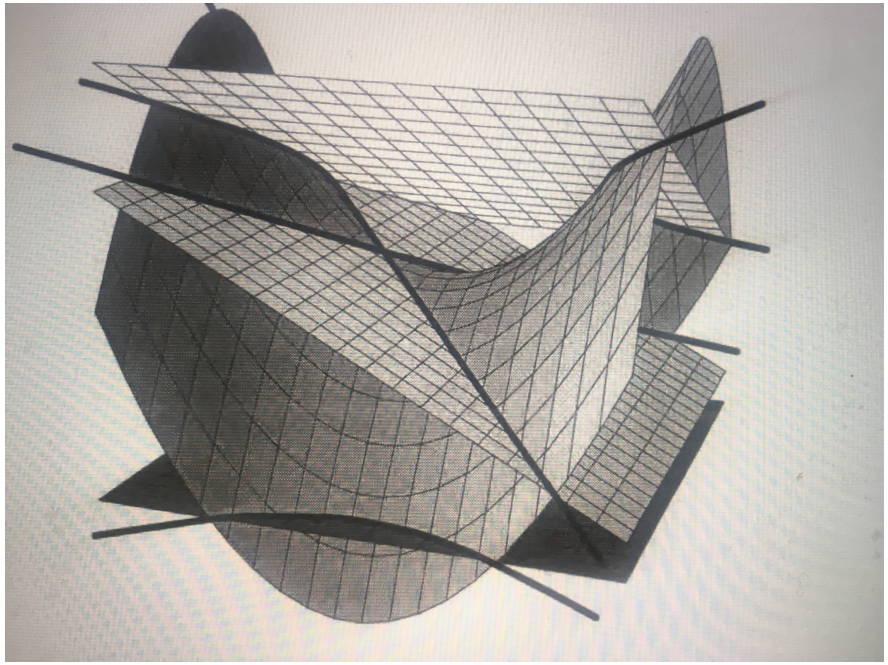

Figure: Smooth hyperplane sections

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