Math 203A
Last time

Motivation:

- We seek to resolve singularities of $X$:
  - $f : Y \to X$ proper birational
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If $f : X \to Y$ we use blowups to extend to regular $\tilde{f} : \tilde{X} \to Y$. 
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Blowups – outline:

- Simplified discussion for blowup of $\mathbb{A}^2$ at $(0,0)$
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**Terminology:** strict transform, exceptional set
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Blowup of the plane (simplified)

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Strict transform

Let $C$ be a curve through 0 of multiplicity $m$: 

$$f(x, y) = \sum_{i+j=m} a_{ij} x^i y^j + h.o.t.$$

**Question:** what is $\pi^{-1}(C)$?

**Answer:** $\pi^{-1}(C) = m \cdot \ell + \tilde{C}$

$\tilde{C}$ = closure of $\pi^{-1}(C \setminus \{0\}) = \text{strict transform of } C$
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Exceptional line

\[ f^{(in)} = \sum_{i+j=m} a_{ij} x^i y^j = \prod_k (\alpha_k x + \beta_k y) \]
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Remarks:

- \( \pi : \tilde{C} \to C \) birational
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Remarks:

- \( \pi : \tilde{\mathcal{C}} \to \mathcal{C} \) birational
- \( \mathcal{C}, D \) have distinct tangent directions at 0, \( \mathcal{C} \cap D = \{0\} \)
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Remarks:

- \( \pi : \tilde{C} \to C \) birational
- \( C, D \) have distinct tangent directions at 0, \( C \cap D = \{0\} \)

\[ \tilde{C} \cap \tilde{D} = \emptyset \implies \text{strict transforms separated} \]
Figure: Blowing up the plane
Example: Node:

\[ C = \{ y^2 = x^2(x + 1) \} \subset \mathbb{A}^2. \]
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Strict transform

\[(uv)^2 = u^2(u + 1)\]

One more blowup resolves the singularity.
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Strict transform

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One more blowup resolves the singularity.
Blowup of $\mathbb{A}^n$ at $p = (0, \ldots, 0)$

$\tilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$
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Remark: $L$ line spanned by $(a_1, \ldots, a_n)$

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- $\tilde{L} \cap E = \{[a_1 : \ldots : a_n]\}$
Affine cover

\[ \mathcal{A}^n = \bigcup_i U_i, \quad U_i = \{(x, y) : y_i \neq 0\} \]
Affine cover

\[ \widetilde{\mathbb{A}}^n = \bigcup_i U_i, \quad U_i = \{(x, y) : y_i \neq 0\} \]

\[ U_1 \simeq \mathbb{A}^n, \quad (x, y) \mapsto \left( x_1, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1} \right) \]
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Example: When \( n = 2 \),

\[ U_1 \simeq \mathbb{A}^2, \quad (x_1, y_2) \mapsto (x_1, x_1y_2) \]

studied previously.
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General construction

- Data

\[ X \subset \mathbb{A}^n, \; Y = \{f_1 = \ldots = f_r = 0\} \subset X, \; U = X \setminus Y \]
General construction

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\[ X \subset \mathbb{A}^n, \quad Y = \{ f_1 = \ldots = f_r = 0 \} \subset X, \quad U = X \setminus Y \]

Construction

\[ f : X \rightarrow \mathbb{P}^{r-1}, \quad f(x) = [f_1(x) : \ldots : f_r(x)] \text{ defined over } U \]
General construction

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► Graph

\[ \Gamma_f \subset U \times \mathbb{P}^{r-1} \]
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- \[ \pi : \tilde{X} \to X \text{ induced by } \Gamma_f \subset X \times \mathbb{P}^{r-1} \to X \]
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Equations for the blowup

Lemma

\[ \tilde{X} \subset \{ (x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \} \]

Remark: This recovers \( \tilde{\mathbb{A}}^n \) for \( f_i(x) = x_i, X = \mathbb{A}^n, Y = \{0\} \).

BEWARE!!! Equality may not hold above.
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\( F, \tilde{F} \) are isomorphisms so

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General construction

Several methods:

- $X$ variety, $Y \subset X$ closed, $X = \bigcup U_i$ affine cover

- $X$ projective – via graphs as in affine case.
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Example: del Pezzo surfaces

Blowups of $\mathbb{P}^2$ at $n \leq 8$ general points.
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Exercise: Blowup of $\mathbb{P}^2$ at 1 point $\simeq$ blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 2 points.
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*Figure:* Pasquale del Pezzo
Example: Cremona transformation

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Figure: Luigi Cremona