Math 203A

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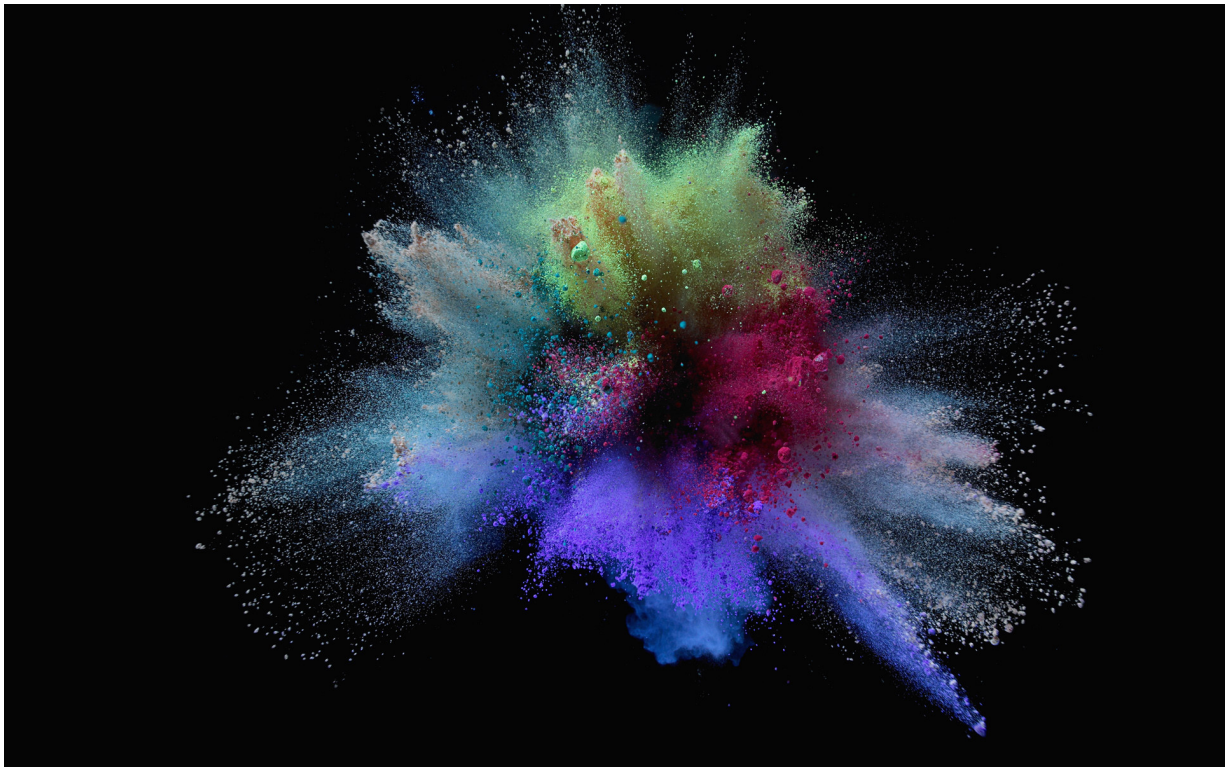
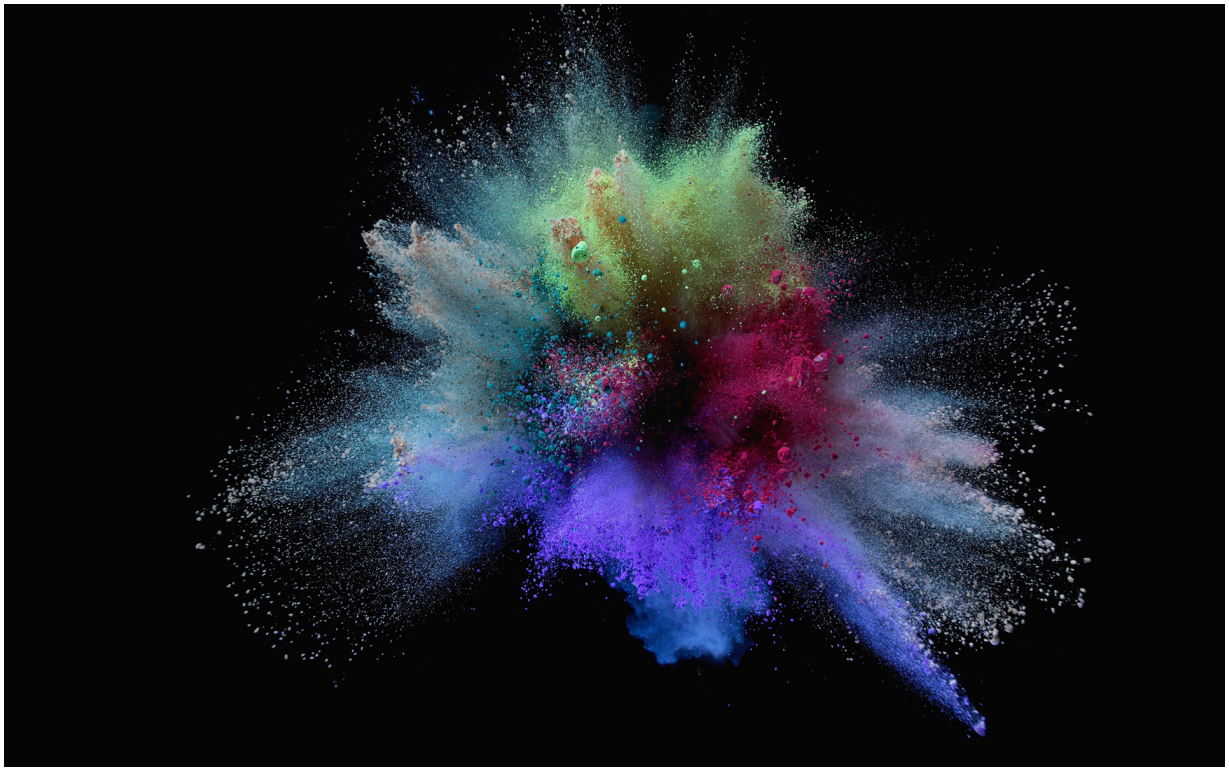


Blowup=Éclatement (French)



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$$\widetilde{C} \cap \widetilde{D} = \emptyset \implies \text{strict transforms separated}$$

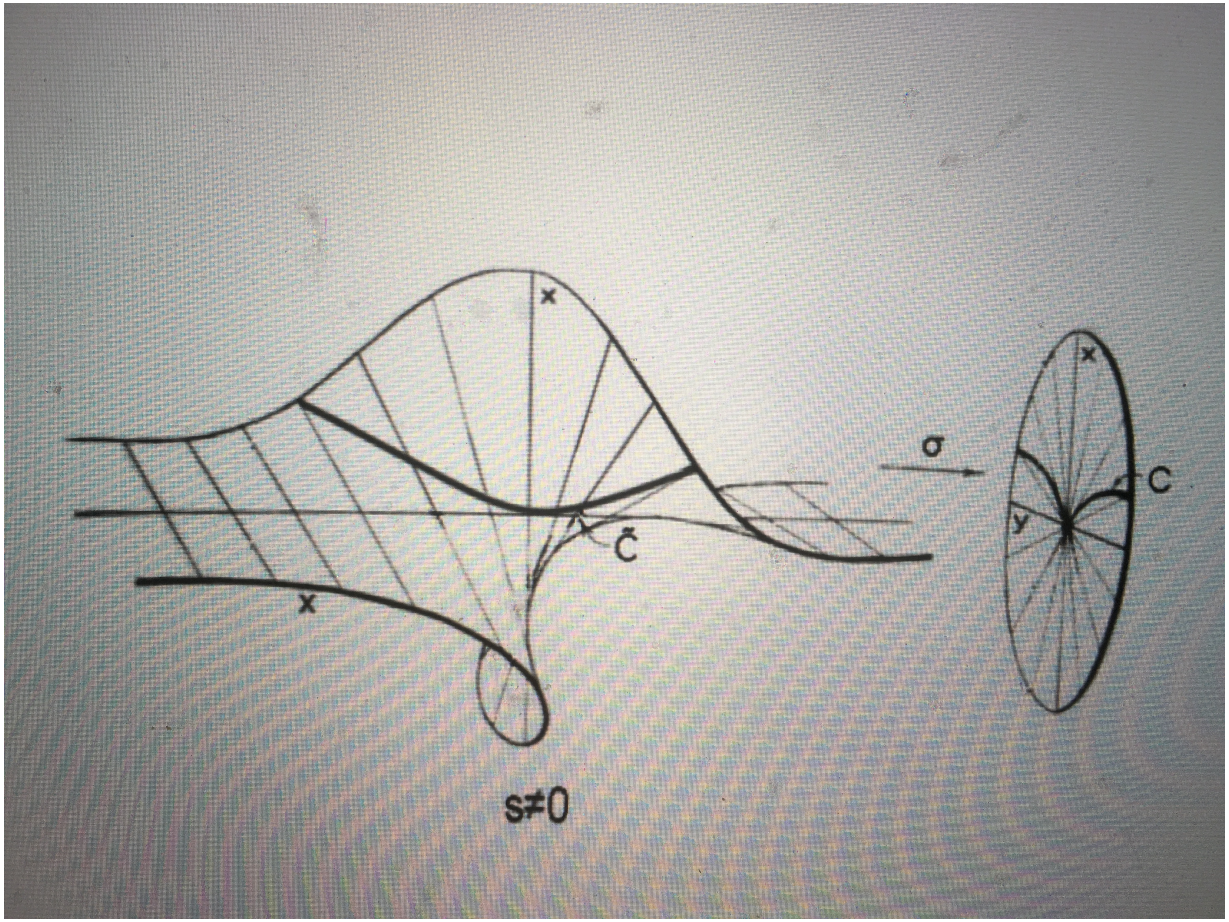

Figure: Blowing up the plane

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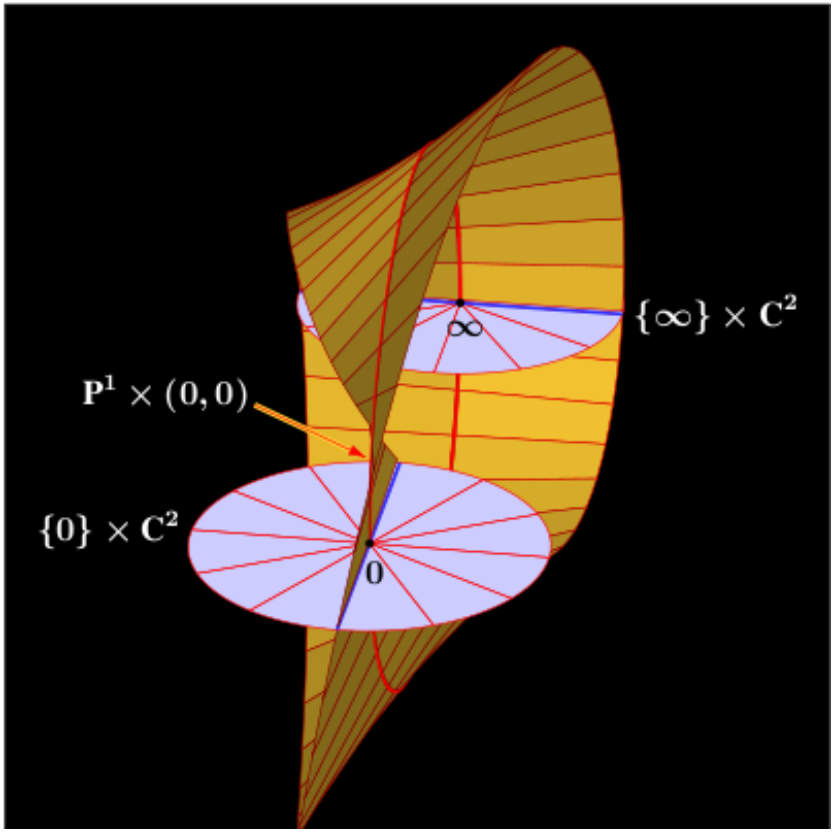

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Figure: Pasquale del Pezzo

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Figure: Luigi Cremona