Math 203A

September 30, 2019

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Figure: David Hilbert

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Definition

A topological space X is reducible if $X = X_1 \cup X_2$ for two proper closed subsets X_1 and X_2.

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Two distinct irreducible polynomials f and g in $k[X, Y]$ have only finitely many common roots.

Since $Z(\mathfrak{p})$ is irreducible, $Z(\mathfrak{p})$ is a point (a, b) . Thus $\mathfrak{p} = (X - a, Y - b)$.

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Figure: Emmy Noether

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- Terminology: curve, surface, threefold, etc.

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Remark: This is true if X is an affine algebraic set in the Zariski topology.

II. Functions and morphisms of algebraic sets

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$$A(X) = k[x_1, \ldots, x_n], \;\; K(X) = k(x_1, \ldots, x_n).$$

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- The quotient field $K(X)$ is called the field of rational functions on X.
- Each rational function can be represented as

$$\phi = \frac{f}{g}$$

- If $\phi = \frac{f}{g} = \frac{f'}{g'}$ then

$$fg' - f'g \in I(X)$$

- Rational functions are partially defined over open subsets

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Let $U \subset X$ be open. We define regular functions on U.

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