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Figure: David Hilbert
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Proof

\[ j = (1): \]

\[ f_i \in \mathbb{R}. \]

\[ t_N \text{ the highest power of } t \text{ occurring in the } g_i. \]
Proof

Hence \( j = (1) \):

\[
1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots
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with \( f_i \in i \).
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Proof

In $k[x_1, \ldots, x_n, t]/(ft - 1)$ we have

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Irreducibility

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Definition
A topological space $X$ is reducible if $X = X_1 \cup X_2$ for two proper closed subsets $X_1$ and $X_2$. 
Irreducibility

Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.
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Let $X$ be irreducible.

- $U$ and $V$ are nonempty open subsets of $X$, then $U \cap V \neq \emptyset$. 

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An irreducible affine algebraic set is called an affine variety.
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Affine algebraic sets are in 1 -- 1 correspondence with radical ideals.
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How about affine varieties?
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Example: The ideal $\mathfrak{a} = (x^2y - y^2)$ is not prime. In fact,

$$y \cdot (x^2 - y) \in \mathfrak{a}$$

but $y \notin \mathfrak{a}$ and $x^2 - y \notin \mathfrak{a}$. 

We have $Z(\mathfrak{a}) = Z(y) \cup Z(y - x^2)$.

Example: The only proper irreducible closed subsets of $\mathbb{A}^1$ are single points.

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Example: The only proper irreducible closed subsets of \( \mathbb{A}^1 \) are single points.

Example: What are the irreducible closed subsets of \( \mathbb{A}^2 \)? What are the prime ideals of \( k[X, Y] \)?
Let $\mathfrak{p} \subset k[X, Y]$ be prime, $\mathfrak{p} \neq (0), (1)$. 
Let \( p \subset k[X, Y] \) be prime, \( p \neq (0), (1) \).

We claim

- \( p \) is principal generated by one irreducible polynomial \( f \) or
- \( p \) is maximal, \( p = (X - a, Y - b) \) for some \( a, b \in k \).
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Geometrically, the proper subvarieties of \( \mathbb{A}^2 \) are points and irreducible affine curves.
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If $\mathfrak{p} \neq (f)$, pick an element $G \in \mathfrak{p} \setminus (f)$.

Pick an irreducible factor of $G$. Then there is an irreducible polynomial $g \neq f$ with $g \in \mathfrak{p}$. 

Now, $(f, g) \subset \mathfrak{p} = \mathbb{Z}(\mathfrak{p}) = \mathbb{Z}(f, g)$.

Two distinct irreducible polynomials $f$ and $g$ in $\mathbb{K}[X, Y]$ have only finitely many common roots.

Since $\mathbb{Z}(\mathfrak{p})$ is irreducible, $\mathbb{Z}(\mathfrak{p})$ is a point $(a, b)$.

Thus $\mathfrak{p} = (X - a, Y - b)$. 

Pick $F \in p$. Factorize $F$ into product of irreducibles. Thus $p$ contains one irreducible polynomial $f$.

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Now,

$$(f, g) \subset \mathfrak{p} \implies Z(\mathfrak{p}) \subset Z(f, g).$$
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Two distinct irreducible polynomials $f$ and $g$ in $k[X, Y]$ have only finitely many common roots.
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Finiteness conditions

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Figure: Emmy Noether
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Let $X$ be a Noetherian topological space. Then $X$ can be written as finite union of irreducible closed subsets

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Algebraically, any radical ideal \( a \) is intersection of prime ideals

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Continuing this construction, one arrives at an infinite chain $X \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq \ldots$, contradicting $X$ is Noetherian.
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Definition
An irreducible Noetherian topological space $X$ has dimension $n$ if

- there is a descending chain of closed irreducible subsets $X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \neq \emptyset$,
- and any other chain has length at most or equal to $n$.

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- Terminology: curve, surface, threefold, etc.
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- $\mathbb{A}^1 \supseteq f^{-1}(X_1) \supseteq f^{-1}(X_2) \neq \emptyset$.

- $f^{-1}(X_i)$ is a proper closed subset of $\mathbb{A}^1$, hence finite.

- Thus $X_i$ is one point.
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▶ This is false.

Set $X = \{a, b\}$. Give $X$ the topology whose closed sets are $\emptyset$, $\{a\}$, $X$. $X$ is irreducible of dimension 1, while $U = \{b\}$ is a dense open set of dimension 0.

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II. Functions and morphisms of algebraic sets
Coordinate rings

- We wish to define regular functions on affine varieties.
Coordinate rings

▶ We wish to define regular functions on affine varieties
  ▶ holomorphic functions, differentiable functions etc.

\[ \mathcal{A}(X) = \mathbb{K}[x_1, \ldots, x_n]/I(X) = \text{integral domain} \]

▶ Any \( f \in \mathcal{A}(X) \) gives a polynomial function \( f : X \to \mathbb{K} \)

▶ This is independent of choices \( f_1, f_2 \) in \( \mathcal{A}(X) = \Rightarrow f_1 - f_2 \in I(X) = \Rightarrow f_1|_X = f_2|_X. \)
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Let $U \subset X$ be open. We define regular functions on $U$. 
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**Strategy:** A regular function on $U$ must be regular at each $p \in U$. 

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\text{Local ring } O_{X, p} = \left\{ f, g : g(p) \neq 0, f, g \in \mathcal{A}(X) \right\} \subset K(X).
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Beware: this is trickier than it looks!!!

Not all regular functions on $U$ can be expressed globally as quotients of two polynomials!!!!

Example: Let $X = \{xw - yz = 0\} \subset A^4$. Let $U = \{y \neq 0 \text{ or } w \neq 0\}$.

The function $\phi = \begin{cases} x & \text{for } y \neq 0 \\ z & \text{for } w \neq 0 \end{cases}$ is well-defined and regular on $U$. It is not a global quotient of two polynomials.
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