Math 203A

October 2, 2019
Last time

We studied:

- affine varieties $X \subset \mathbb{A}^n$
- irreducibility, dimension
- functions defined over them
- coordinate rings
- regular functions over open subsets
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Regular functions on open subsets

\[ U \subset X \text{ open.} \]
Regular functions on open subsets

$U \subset X$ open.

Let $p \in X$. The local ring

$$\mathcal{O}_{X,p} = \left\{ \frac{f}{g} : g(p) \neq 0, f, g \in A(X) \right\} \subset K(X).$$
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These are the regular functions at $p$. 
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\textbf{Example:} Let $X = \{xw - yz = 0\} \subset \mathbb{A}^4$. Let

$$U = \{y \neq 0 \text{ or } w \neq 0\}.$$
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The function

$$\phi = \begin{cases}  
\frac{x}{y} & \text{for } y \neq 0 \\
\frac{z}{w} & \text{for } w \neq 0
\end{cases}$$

is well-defined and regular on $U$. 

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Regularity is local

Lemma
Let $U \subset X \subset \mathbb{A}^n$ be open. Let

$$\phi : U \to k$$

be a set theoretic map.

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Remark: A well-defined function on $U$ is regular if it can be written locally as a quotient.
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- **Forward:** We have $\phi = \frac{f}{g}$ with $g(p) \neq 0$. 
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This is open in $U$ and contains $p$. 

**Converse:** to each $\phi = \frac{f}{g}$ in $V$, associate $f \frac{1}{g} \in K(X)$. 
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    \implies fg' = f'g \text{ in } A(X)
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\end{itemize}
- Check: this is independent of choices.

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Distinguished open sets

Let \( f \in A(X) \), \( f \neq 0 \).
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- The open sets

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X_f = X \setminus Z(f) = \{ p \in X : f(p) \neq 0 \}
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are called distinguished.
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- \( X_f \) form a basis for the Zariski topology:

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X \setminus U = Z(\{f_i\}) \implies U = \bigcup_i X_{f_i}.
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Lemma

Regular functions on \( X_f \) are global quotients:

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Lemma

Regular functions on $X_f$ are global quotients:

$$\mathcal{O}_X(X_f) = A(X)_f = \left\{ \frac{g}{f^r}, g \in A(X) \right\}.$$
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In particular,

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\mathcal{O}_X(X) = A(X).
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- ideal of denominators

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$$j = \{g \in A(X) : g\phi \in A(X)\}$$

- We show $f' \in j$ for some $r$

Note $h(p) \neq 0 \Rightarrow p \notin Z(j) \Rightarrow Z(j) \subset Z(f)$

$IZ(f) \subset IZ(j) \Rightarrow f \in \sqrt{j} = \Rightarrow f' \in j$. 
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- Take $p \in X$,

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Removable singularity theorem

Lemma

Let $U = \mathbb{A}^2 \setminus \{(0, 0)\}$. 

Hartogs' theorem: All holomorphic functions on $\mathbb{C}^n \setminus \{0\}$ for $n \geq 2$ extend across the origin.
Removable singularity theorem

Lemma

Let \( U = \mathbb{A}^2 \setminus \{(0, 0)\} \). Then \( \mathcal{O}_{\mathbb{A}^2}(U) = k[x, y] \).
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*All functions on $U$ extend across the origin!*

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Let $f$ be regular on $U$. 

\[ f(x, y) = p(x, y) x^n = q(x, y) y^m \]

which gives $n = m = 0$ and $f$ is a polynomial.
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Goal: New structures
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- functions on affine sets $\rightarrow$ sheaves
Goal: New **structures**

- functions on affine sets → **sheaves**

- affine varieties → **ringed spaces**
Goal: New structures

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Bonus:
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- affine varieties → ringed spaces

Bonus:
- sheaves → morphisms
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Bonus:

- sheaves $\rightarrow$ morphisms

- ringed spaces $\rightarrow$ varieties
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**Definition**

Let $X$ be a topological space. A *presheaf* $\mathcal{F}$ of rings over $X$ is the datum of
Presheaves

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Let $X$ be a topological space. A presheaf $\mathcal{F}$ of rings over $X$ is the datum of

- an \textbf{assignment} of a ring

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for each $U \subset X$ open;
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$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

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- $s \in \mathcal{F}(U) = \Gamma(U, \mathcal{F})$ are called sections of $\mathcal{F}$
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Notation:

- $\rho_{V,U}(s) = s|_U$ for $s \in \mathcal{F}(V)$
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Sheaves

Definition
A presheaf $\mathcal{F} \to X$ is said to be a sheaf provided that for all open covers

$$U = \bigcup_i U_i, \quad U_{ij} = U_i \cap U_j$$
Sheaves

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Lemma

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