Math 203A

October 2, 2019

Last time

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Lemma

Let $U \subset X \subset \mathbb{A}^n$ be open. Let

$$\phi : U \to k$$

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- $s_i \in \mathcal{F}(U_i)$ with
- $s_i|_{U_{ij}} = s_j|_{U_{ij}}$

there exists a unique $s \in \mathcal{F}(U)$ with

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