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- **Equivalence**
  - affine varieties
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- We wish to define **arbitrary varieties**.
Roadmap:

affine varieties $\mapsto$ ringed spaces
Roadmap:

affine varieties $\mapsto$ ringed spaces $\mapsto$ abstract affine varieties
Roadmap:

affine varieties  \mapsto ringed spaces  \mapsto abstract affine varieties  \mapsto prevarieties
Roadmap:

affine varieties $\mapsto$ ringed spaces $\mapsto$ abstract affine varieties $\mapsto$

prevarieties $\mapsto$ varieties
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Abstract affine varieties

- We need a coordinate-free definition of affine varieties
Abstract affine varieties

- We need a \textit{coordinate-free} definition of affine varieties
- This will make it easier to \textit{glue} affine varieties
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Definition

An abstract affine variety \((X, \mathcal{O}_X)\) is a ringed space
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▶ \(\mathcal{O}_X\) is a sheaf of \(k\)-valued functions
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\textbf{Definition}

An \textit{abstract affine variety} \((X, \mathcal{O}_X)\) is a \textit{ringed space}

- \(\mathcal{O}_X\) is a sheaf of \(k\)-valued functions
- \(X\) is irreducible as a topological space
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Basic open sets are affine

Lemma

Let $X \subset \mathbb{A}^n$ be an affine variety. The basic open set

$$X_f = \{x \in X : f(x) \neq 0\}$$

is an abstract affine variety.
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Proof:

$$Z \subset \mathbb{A}^{n+1}, \quad Z = \{(x, t) : x \in X, tf(x) - 1 = 0 \}.$$
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- $\pi : Z \to X_f, \ (x, t) \mapsto x$
- $\tau : X_f \to Z, \ x \mapsto \left( x, \frac{1}{f(x)} \right)$
Example:

- $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to $xy - 1 = 0$ in $\mathbb{A}^2$
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Prevarieties

▶ We glue affine varieties to get new objects
Prevarieties

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**Definition**

A prevariety is a ringed space \((X, \mathcal{O}_X)\) such that

- \(X\) is irreducible

Remarks:

- An open set with \((U_i, \mathcal{O}_X|_{U_i})\) abstract affine variety is called affine open.
- This is similar to the definition of manifolds.
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**Definition**

A prevariety is a ringed space \((X, \mathcal{O}_X)\) such that
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- \(X\) admits a finite cover by open sets \(X = \bigcup U_i\) such that \((U_i, \mathcal{O}_X|_{U_i})\) is an abstract affine variety.

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\end{itemize}
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Construction of prevarieties – Gluing

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- How about gluing **prevarietites**?
Gluing data

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Construct the prevariety $X$: 

$X$ is the disjoint union $X_1, X_2$ modulo the equivalence $p \equiv f(p), p \in U_1$. $X$ carries the quotient topology induced by $\equiv$.

Sheaf $O_X$: $O_X(U) = \{(g_1, g_2), g_1 \in O_{X_1}(U \cap X_1), g_2 \in O_{X_2}(U \cap X_2) : g_1|_{U \cap U_1} = g_2|_{U \cap U_2}\}$.

Check: $O_X$ is a sheaf, $X$ is irreducible, every point of $X$ has an affine neighborhood.
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More gluing

Lemma

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Examples

\[ X_1 = X_2 = \mathbb{A}^1, \quad U_1 = U_2 = \mathbb{A}^1 \setminus \{0\} \]
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$X_1 = X_2 = \mathbb{A}^1$, $U_1 = U_2 = \mathbb{A}^1 \setminus \{0\}$

$f : U_1 \to U_2$, $x \mapsto 1/x$. 

Projective line $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ where $\infty = 0/0$. 

Affine line with double origin.
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Affine line with double origin.
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Lemma (Gluing of morphisms)
Let $f : X \to Y$ be a set-theoretic map of prevarieties.
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Lemma (Gluing of morphisms)

Let $f : X \to Y$ be a set-theoretic map of prevarieties. Let $\{U_i\}$ and $\{V_i\}$ be open covers of $X$ and $Y$. 
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$$f_i = f|_{U_i} : U_i \to V_i.$$
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**Lemma (Gluing of morphisms)**

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*Then* $f$ *is a morphism iff* $f_i$ *are morphisms.*
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Then $f$ is a morphism iff $f_i$ are morphisms.
If \( f_i \) are morphisms, \( f_i \) continuous, so \( f \) is continuous.
If $f_i$ are morphisms, $f_i$ continuous, so $f$ is continuous

We show

$$\phi \in \mathcal{O}_Y(V) \implies f^*\phi \in \mathcal{O}_X(f^{-1}(V))$$
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- Since $\mathcal{O}_X$ is sheaf, $f_i^*\phi_i$ glue to a unique regular section

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Any morphism

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Roadmap:
Additional constructions: products

**Goal:** if $X$, $Y$ are prevarieties, we define $X \times Y$ as a prevariety.
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$$(i) + (iii) \implies (iv), \ (iv) + (iii) \implies (v)$$
1. Affine case

Let

\[ X \subset \mathbb{A}^n, \ Y \subset \mathbb{A}^m \]

given by polynomials

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affine varieties $\mapsto$ ringed spaces $\mapsto$ abstract affine varieties $\mapsto$

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