Math 203A

October 16, 2019

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Lemma

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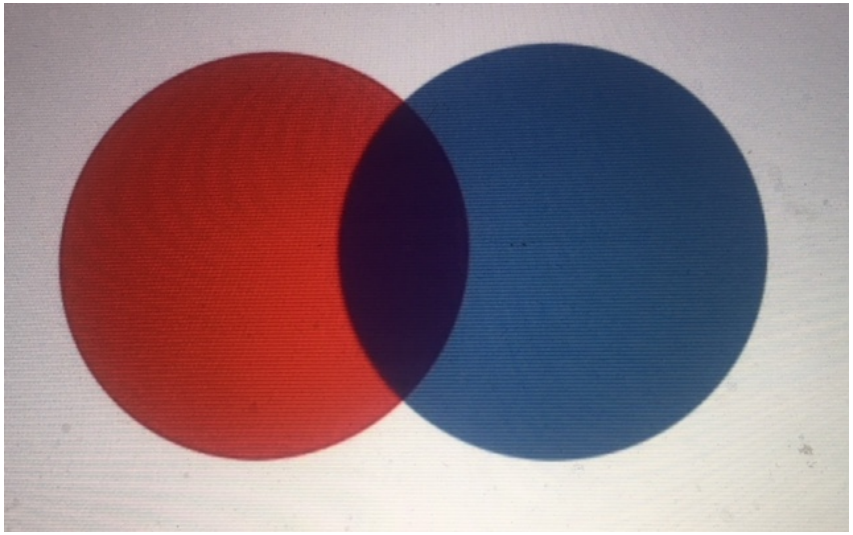

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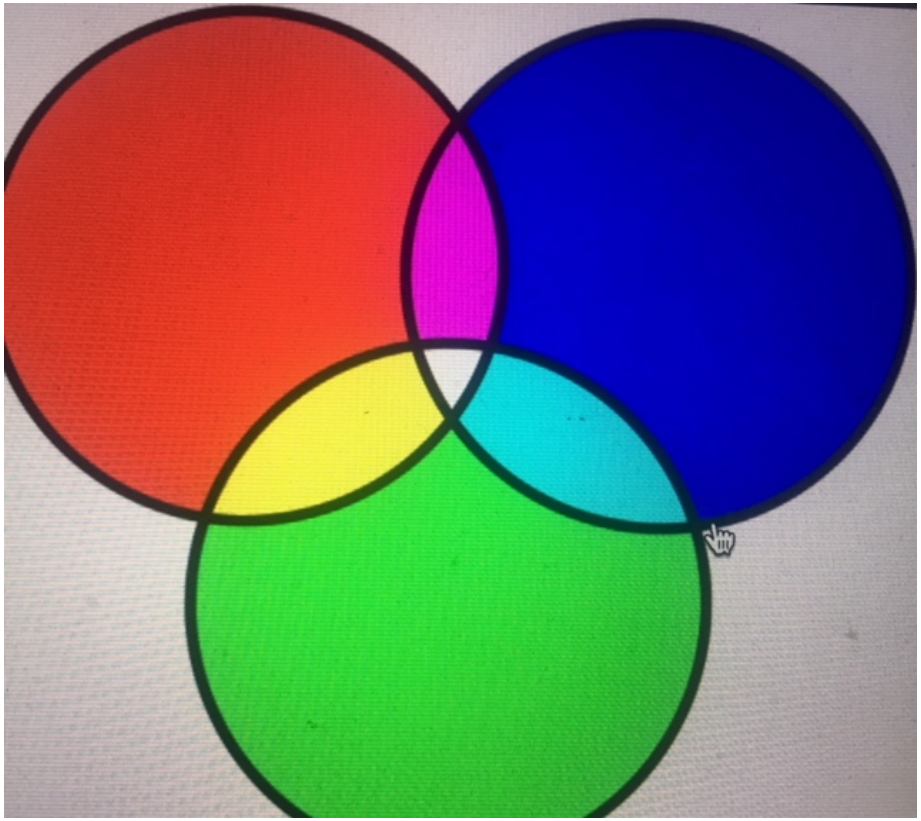

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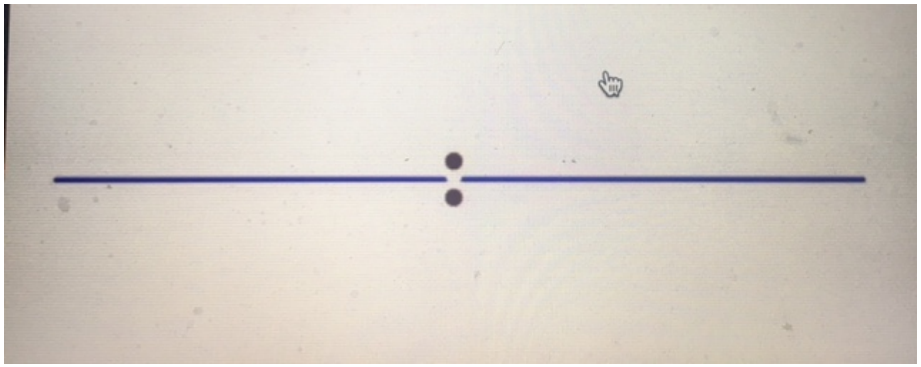

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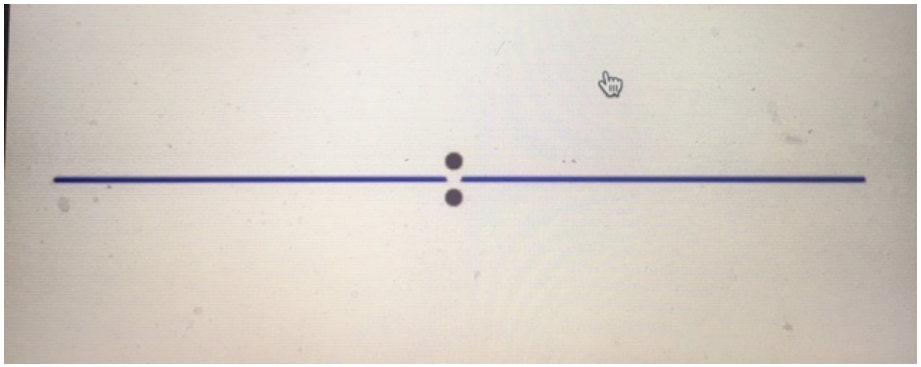

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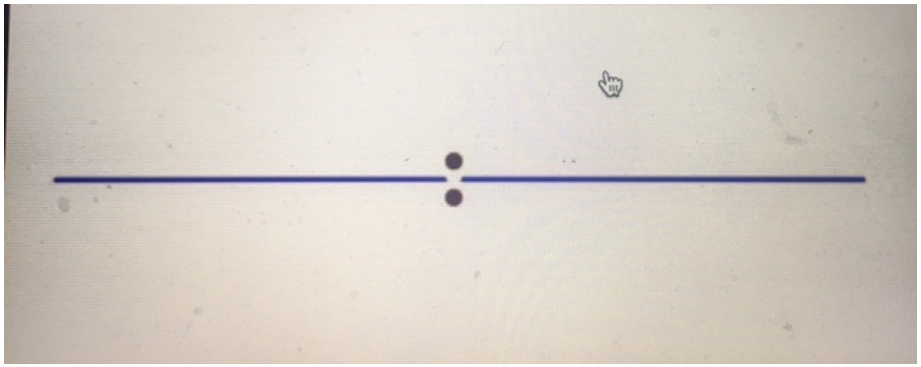

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