**Math 203, Problem Set 2. Due Friday, October 18.**

*For this problem set, you may assume that the ground field is $k = \mathbb{C}$.*

**1.** An algebraic set $Z \subset \mathbb{A}^2$ defined by an irreducible polynomial $f$ of degree 2 is called an irreducible conic.

(i) Show that any irreducible conic can be written in the form

$$Y - X^2 = 0 \text{ or } XY - 1 = 0$$

after an affine change of coordinates in $\mathbb{A}^2$.

*Remark:* An affine change of coordinates taking $(x, y)$ into $(X, Y)$ is a transformation of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b},$$

where $A$ is a $2 \times 2$ invertible matrix and $\mathbf{b} \in \mathbb{A}^2$ is a vector.

*Hint:* Write $f(x, y) = a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6$, and complete the square.

(ii) Let $Z_1$ and $Z_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that $Z_1$ and $Z_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

**2.** Find the coordinate rings of the following affine algebraic sets and decide which of the following algebraic sets are isomorphic, and which ones are not:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^3$.

**3.** (*Hartogs theorem and quasi-affine algebraic sets.*) Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0, 0)\}$ is not isomorphic to an affine algebraic set.

*Hint:* Argue by contradiction. Using your knowledge about the regular functions on $X$, what can you say about the inclusion $\iota : X \to \mathbb{A}^2$?

**4.** Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with $n$ elements in $\mathbb{A}^1$.

(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take $X$ to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

(ii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

*Hint:* Assume that

$$\Phi : A(X) \to k\left[t, t^{-1}\right]$$

is an isomorphism. Observe that $X_i$ are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers $m_i$. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of $i$.

**5.** Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that $Z_n$ and $W_n$ are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \to Z_n, \;\; f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n, \;\; g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and $W_n$ where $g$ becomes an isomorphism.

(ii) Using $(i)$, explain why $Z_n$ and $W_n$ are birational.

(iii) Assume that there exists an isomorphism

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part $(i)$ and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) (*Optional.*) Repeat the argument above without the assumption that $h$ sends the origin to itself. You may need to prove a stronger version of Problem 4.

**6.** (*Quotients.*) Taking quotients in algebraic geometry is subtle. We will explain how to take quotients by finite groups.

Let $X$ be an affine variety, and let $G$ be a finite group. Assume that $G$ acts on $X$ algebraically, i.e. that for every $g \in G$, we are given a morphism $g : X \to X$ (denoted by the same letter for simplicity of notation), such that

$$(gh)(p) = g(h(p))$$

for all $g, h \in G$ and $p \in X$.

(i) Let $g \in G$ act on the coordinate rings $A(X)$ via

$$f \mapsto f^g \text{ with } f^g(p) = f(g(p)).$$

Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all $G$-invariant functions on $X$. Show that $A(X)^G$ is a finitely generated $k$-algebra.

(ii) By (i), there is an affine variety $Y$ with coordinate ring $A(X)^G$, together with a morphism

$$\pi : X \to Y$$

determined by the inclusion

$$A(X)^G \hookrightarrow A(X).$$

Show that $Y$ can be considered as the quotient of $X$ by $G$, denoted $X/G$, in the following sense: if $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(iii) Let

$$\mu_n = \left\{ \exp\left(\frac{2\pi i k}{n}\right), k \in \mathbb{Z} \right\}$$

be the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Describe $\mathbb{C}/\mu_n$ and $\mathbb{C}^2/\mu_n$ as affine algebraic sets.