For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. An algebraic set $Z \subset \mathbb{A}^2$ defined by an irreducible polynomial $f$ of degree 2 is called an irreducible conic.

   (i) Show that any irreducible conic can be written in the form
   
   \[ Y - X^2 = 0 \text{ or } XY - 1 = 0 \]
   
   after an affine change of coordinates in $\mathbb{A}^2$.

   \text{Remark:} An affine change of coordinates taking $(x, y)$ into $(X, Y)$ is a transformation of the form
   
   \[
   \begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + b,
   
   \]
   
   where $A$ is a $2 \times 2$ invertible matrix and $b \in \mathbb{A}^2$ is a vector.

   \text{Hint:} Write $f(x, y) = a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6$, and complete the square.

   (ii) Let $Z_1$ and $Z_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that $Z_1$ and $Z_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

2. Find the coordinate rings of the following affine algebraic sets and decide which of the following algebraic sets are isomorphic, and which ones are not:

   (i) $\mathbb{A}^1$
   (ii) $\mathbb{Z}(xy) \subset \mathbb{A}^2$
   (iii) $\mathbb{Z}(x^2 + y^2) \subset \mathbb{A}^2$
   (iv) $\mathbb{Z}(x^2 - y^5) \subset \mathbb{A}^2$
   (v) $\mathbb{Z}(y - x^2, z - x^3) \subset \mathbb{A}^3$.

3. (Hartogs theorem and quasi-affine algebraic sets.) Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0, 0)\}$ is not isomorphic to an affine algebraic set.

   \text{Hint:} Argue by contradiction. Using your knowledge about the regular functions on $X$, what can you say about the inclusion $\iota : X \to \mathbb{A}^2$?

4. Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with $n$ elements in $\mathbb{A}^1$. 
(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take $X$ to be the affine algebraic set given by the equations
\[ X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1. \]

(ii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

*Hint:* Assume that $\Phi : A(X) \to k[t, t^{-1}]$ is an isomorphism. Observe that $X_i$ are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers $m_i$. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of $i$.

5. Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:
\[ Z_n = \mathbb{Z}(y^n - x^{n+1}) \]
and
\[ W_n = \mathbb{Z}(y^n - x^n(x + 1)). \]
Show that $Z_n$ and $W_n$ are birational but not isomorphic.

(i) Show that
\[ f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1}) \]
is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets
\[ \mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}. \]
Similarly, show that
\[ g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t). \]
is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and $W_n$ where $g$ becomes an isomorphism.

(ii) Using (i), explain why $Z_n$ and $W_n$ are birational.

(iii) Assume that there exists an isomorphism
\[ h : Z_n \to W_n \]
such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets
\[ Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}. \]
Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$. 

(iv) *(Optional.)* Repeat the argument above without the assumption that \( h \) sends the origin to itself. You may need to prove a stronger version of Problem 4.

6. *(Quotients.)* Taking quotients in algebraic geometry is subtle. We will explain how to take quotients by finite groups.

Let \( X \) be an affine variety, and let \( G \) be a finite group. Assume that \( G \) acts on \( X \) algebraically, i.e. that for every \( g \in G \), we are given a morphism \( g : X \to X \) (denoted by the same letter for simplicity of notation), such that

\[
(gh)(p) = g(h(p))
\]

for all \( g, h \in G \) and \( p \in X \).

(i) Let \( g \in G \) act on the coordinate rings \( A(X) \) via

\[
f \mapsto f^g \quad \text{with} \quad f^g(p) = f(g(p)).
\]

Let \( A(X)^G \) be the subalgebra of \( A(X) \) consisting of all \( G \)-invariant functions on \( X \). Show that \( A(X)^G \) is a finitely generated \( k \)-algebra.

(ii) By (i), there is an affine variety \( Y \) with coordinate ring \( A(X)^G \), together with a morphism

\[
\pi : X \to Y
\]

determined by the inclusion

\[
A(X)^G \hookrightarrow A(X).
\]

Show that \( Y \) can be considered as the quotient of \( X \) by \( G \), denoted \( X/G \), in the following sense: if \( p, q \in X \) then \( \pi(p) = \pi(q) \) if and only if there is a \( g \in G \) such that \( g(p) = q \).

(iii) Let

\[
\mu_n = \left\{ \exp \left( \frac{2\pi i k}{n} \right), k \in \mathbb{Z} \right\}
\]

be the group of \( n \)-th roots of unity. Let \( \mu_n \) act on \( \mathbb{C}^m \) by multiplication in each coordinate. Describe \( \mathbb{C}/\mu_n \) and \( \mathbb{C}^2/\mu_n \) as affine algebraic sets.