Math 203, Problem Set 4. Due Friday, November 1.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Let $V$ be a finite dimensional vector space and let $\omega \in \Lambda^2 V$ be such that $\omega \wedge \omega = 0$. Show that $\omega = a \wedge b$ for some vectors $a, b \in V$.

2. The set $X$ of degree $d$ homogeneous polynomials in $n + 1$ variables can be identified with a projective space $\mathbb{P}^N$, by recording the coefficients in some order. What is $N$?

Using the fundamental theorem of elimination theory, show that the set of reducible polynomials form a closed subset of $X$.

Remark: For instance, when $d = n = 2$, the quadratic polynomial $\sum_{i,j=0}^2 a_{ij} z_i z_j$ is reducible iff $\det(a_{ij}) = 0$.

3. Show that $\mathbb{P}^1 \times \mathbb{A}^1$ and $\mathbb{P}^2 \setminus \{x\}$ are neither affine nor projective.

Hint: Compute the ring of regular functions.

4. (Joins.) Let $G(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Show that:

(i) The incidence correspondence $\{(L, P) : P \in L\} \subset G(1, n) \times \mathbb{P}^n$ is closed.

Hint: For part (i), describe the incidence correspondence via the Plucker coordinates of $L$. The formalism of exterior algebra helps here. How would you express the condition $P \in L$?

(ii) If $Z \subset G(1, n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$. We say that $W$ is swept out by the lines in $Z$.

(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting $X$ and $Y$ is a closed subset of $\mathbb{P}^n$. It is called the join $J(X, Y)$ of $X$ and $Y$.

Hint: For (ii), (iii) use the fundamental theorem of elimination theory.

5. (Rational varieties and their birational automorphisms.) The definition of birational isomorphisms given in class extends to the projective category. Two projective varieties $X$ and $Y$ are birational if there are rational maps

$$f : X \dasharrow Y, \quad g : Y \dasharrow X,$$

which are rational inverses to each other. Just as in the affine case, a birational isomorphism $f : X \dasharrow Y$ induces an isomorphism of the fields of rational functions

$$f^* : K(Y) \to K(X)$$

and conversely.

(i) We say that $X$ is rational if $X$ is birational to $\mathbb{P}^n$ for some $n$. Explain that if $X$ is rational, then

$$K(X) \cong k(t_1, \ldots, t_n).$$
(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X$ and $Y$ are rational, then $X \times Y$ is rational.

**Remark:** It is very difficult to determine if a given variety is rational. We have seen that lines, conics in $\mathbb{P}^2$ are rational, while elliptic curves in $\mathbb{P}^2$ are not. Twisted cubics in $\mathbb{P}^3$ are rational. We will prove below that quadrics are rational. Smooth cubic surfaces in $\mathbb{P}^3$ also turn out to be rational. However, most varieties are not rational.

(iii) Show that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (but they are birationally isomorphic). You may want to find two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ which do not intersect.

(iv) The group of automorphisms of the field of fractions in two variables $k(x, y)$ is called the Cremona group. Explain that the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$. Explain that the Cremona involution

$$(x, y) \rightarrow (x^{-1}, y^{-1})$$

extends to an automorphism of $k(x, y)$. What is the corresponding birational involution of $\mathbb{P}^2$? Where is this birational automorphism regular?

(v) More generally, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group, in such a fashion that $-1$ corresponds to the Cremona transformation.

**Remark:** The Cremona group is not yet fully understood (especially when the number of indeterminantes $t_i$ is bigger than 2).

6. (*Quadrics are rational.*) Using the projection from a point to a hyperplane, show that any non-degenerate irreducible quadric $Q \subset \mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$.

You may pick a convenient quadric to make the calculations easier.