Math 203, Problem Set 7. Due Friday, December 6.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. (Tangent cones.) Let $X \subset \mathbb{A}^n$ be an affine variety and let $p \in X$. Let $m$ be the maximal ideal of $\mathcal{O}_{X,p}$. Show that the coordinate ring $A(C_{X,p})$ of the tangent cone of $X$ at $p$ is isomorphic to the graded algebra $\bigoplus_{k \geq 0} m^k/m^{k+1}$.

Hint: Let $i \subset k[x_1, ..., x_n]$ be the ideal of $X$. Show that $k[x_1, ..., x_n]/i \rightarrow \bigoplus_{k \geq 0} m^k/m^{k+1}$ given by $f \mapsto f^{(k)}|_X$ is an isomorphism. We had a similar argument in class for the tangent space.

2. (Resolving curve singularities.) Resolve the following $A_k$ plane curve singularity by subsequent blow-ups $y^2 - x^{k+1} = 0$, $k$ odd.

Remark: We have the following terminology on isolated “simple” singularities of hypersurfaces in $\mathbb{A}^{n+2}$:

- type $A_k$: $x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $D_k$: $x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_6$: $x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_7$: $x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_8$: $x^5 + y^3 + z_1^2 + \ldots + z_n^2 = 0$.

(The names suggest a connection with the Weyl groups of type $A, D, E$.)

3. (Cremona transformations.) Consider the Cremona birational automorphism of $\mathbb{P}^2$ given by $C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1]$.

Let $\widetilde{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$ where $C$ is undefined. Show that

(i) Show that $C$ extends to an isomorphism $\widetilde{C} : \widetilde{\mathbb{P}}^2 \rightarrow \widetilde{\mathbb{P}}^2$.

(ii) Let $E_1, E_2, E_3$ be the exceptional lines for the blowup, and let $L_{ij}$ be the strict transform of the line through $P_i$ and $P_j$. Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under $\widetilde{C}$?

Hint: Show that the equations of the blowup $\widetilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ are given by $x_0y_0 = x_1y_1 = x_2y_2$.

It may help to find the equations of the exceptional lines $E_1, E_2, E_3$ and those of the strict transforms $L_{23}, L_{12}, L_{13}$. 


4*. (Del Pezzo surfaces.) From Gathmann, read (but do not hand in) the proof that the blowup of \( \mathbb{P}^2 \) at 2 points is isomorphic to the blowup of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at 1 point.

5. (Degree of the Segre embedding.) Show that the Segre embedding \( \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} \) has degree \( \binom{n+m}{n} \).

6. (Arithmetic genus.) Let \( X \subset \mathbb{P}^n \) be a projective variety with Hilbert polynomial \( \chi_X \). Define the arithmetic genus of \( X \) to be

\[
p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1).
\]

(i) Show that the genus of \( \mathbb{P}^n \) is zero.

(ii) If \( X \) is a hypersurface of degree \( d \) in \( \mathbb{P}^n \), show that \( p_a(X) = \binom{d-1}{n} \). In particular, a cubic in \( \mathbb{P}^2 \) has genus 1.

(iii) If \( X \) is a complete intersection of two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) then

\[
p_a(X) = \frac{1}{2} ab(a + b - 4) + 1.
\]

In particular, intersection of two quadrics in \( \mathbb{P}^3 \) has genus 1.

*Hint:* For (ii) and (iii), you will need to write down suitable exact sequences.