Math 203, Problem Set 7. Due Friday, December 6.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. (*Tangent cones.*) Let $X \subset \mathbb{A}^n$ be an affine variety and let $p \in X$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p}$. Show that the coordinate ring $A(C_{X,p})$ of the tangent cone of X at p is isomorphic to the graded algebra $\bigoplus_{k\geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$.

Hint: Let $\mathfrak{i} \subset k[x_1, \ldots, x_n]$ be the ideal of X. Show that

$$k[x_1, \ldots, x_n]/\mathfrak{i}^{in} \to \bigoplus_{k\geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$$

given by $f \mapsto f^{(k)}|_X$ is an isomorphism. We had a similar argument in class for the tangent space.

2. (*Resolving curve singularities.*) Resolve the following A_k plane curve singularity by subsequent blow-ups

$$y^2 - x^{k+1} = 0, \;\; k \text{ odd}.$$

Remark: We have the following terminology on isolated "simple" singularities of hypersurfaces in $\mathbb{A}^{n+2}$:

- type A_k: $x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type D_k: $x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_6: $x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_7: $x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_8: $x^5 + y^3 + z_1^2 + \ldots + z_n^2 = 0$.

(The names suggest a connection with the Weyl groups of type A, D, E.)

3. (*Cremona transformations.*) Consider the Cremona birational automorphism of $\mathbb{P}^2$ given by

$$C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1].$$

Let $\widetilde{\mathbb{P}^2}$ be the blowup of $\mathbb{P}^2$ at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$ where C is undefined. Show that

(i) Show that C extends to an isomorphism

$$\widetilde{C} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}.$$

(ii) Let E_1, E_2, E_3 be the exceptional lines for the blowup, and let L_{ij} be the strict transform of the line through P_i and P_j. Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under $\widetilde{C}$?

Hint: Show that the equations of the blowup $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are given by

$$x_0y_0 = x_1y_1 = x_2y_2.$$

It may help to find the equations of the exceptional lines E_1, E_2, E_3 and those of the strict transforms L_{23}, L_{12}, L_{13}.

4*. (*del Pezzo surfaces.*) From Gathmann, read (but do not hand in) the proof that the blowup of $\mathbb{P}^2$ at 2 points is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 1 point.

5. (*Degree of the Segre embedding.*) Show that the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ has degree $\binom{n+m}{n}$.

6. (*Arithmetic genus.*) Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert polynomial χ_X. Define the arithmetic genus of X to be

$$p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1).$$

(i) Show that the genus of $\mathbb{P}^n$ is zero.

(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $p_a(X) = \binom{d-1}{n}$. In particular, a cubic in $\mathbb{P}^2$ has genus 1.

(iii) If X is a complete intersection of two surfaces of degree a and b in $\mathbb{P}^3$ then

$$p_a(X) = \frac{1}{2}ab(a+b-4) + 1.$$

In particular, intersection of two quadrics in $\mathbb{P}^3$ has genus 1.

Hint: For (ii) and (iii), you will need to write down suitable exact sequences.