Math 203A - Solution Set 1

Problem 1. Show that the Zariski topology on $\mathbb{A}^2$ is not the product of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

Answer: Clearly, the diagonal
$$Z = \{(x, y) : x - y = 0\} \subset \mathbb{A}^2$$
is closed in the Zariski topology of $\mathbb{A}^2$. We claim this $Z$ is not closed in the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$. Assuming otherwise, the complement $\mathbb{A}^2 \setminus Z$ should be open in the product topology. Therefore, there are nonempty open sets $U, V$ in $\mathbb{A}^1$ such that
$$U \times V \subset \mathbb{A}^2 \setminus Z.$$Now, the sets $U$ and $V$ have finite complements:
$$U = \mathbb{A}^1 \setminus \{p_1, \ldots, p_n\},$$
$$V = \mathbb{A}^1 \setminus \{q_1, \ldots, q_m\}.$$We obtain a contradiction picking
$$z \in \mathbb{A}^1 \setminus \{p_1, \ldots, p_n, q_1, \ldots, q_m\}.$$In this case, we clearly have
$$(z, z) \in U \times V, \text{ while } (z, z) \notin \mathbb{A}^2 \setminus Z.$$□

Problem 2. A topological space $X$ is said to be Noetherian if it satisfies the ascending chain condition on open sets, i.e. any ascending chain of open sets eventually stabilizes.

(i) Check that any subset $Y \subset X$ of a Noetherian space is also Noetherian in the subspace topology.

(ii) Show that $\mathbb{A}^n$ is Noetherian in the Zariski topology. Conclude that any affine algebraic set is Noetherian.

(iii) Show that a Noetherian space is quasi-compact i.e., show that any open cover of a Noetherian space has a finite subcover.

Answer: (i) If $\{U_i\}_{i \geq 1}$ is an ascending chain of open subsets of $Y$, there exists $\{V_i\}_{i \geq 1}$ open subsets of $X$ such that
$$V_i \cap Y = U_i$$for all $i$. Note that $\{V_i\}_{i \geq 1}$ may not be ascending, but let
$$W_i = \bigcup_{k=1}^i V_k.$$
Then $\{W_i\}_{i \geq 1}$ is clearly ascending and

$$W_i \cap Y = (\cup_{k=1}^i V_k) \cap Y = \cup_{k=1}^i (V_k \cap Y) = \cup_{k=1}^i U_k = U_i.$$

Using that $\{W_i\}_{i \geq 1}$ is an ascending chain of open subsets of $X$, and that $X$ is Noetherian, there exists $n_0$ such that $W_n = W_{n_0}$ for all $n \geq n_0$. Therefore,

$$U_n = W_n \cap Y = W_{n_0} \cap Y = U_{n_0}$$

for all $n \geq n_0$, i.e. $\{U_i\}_{i \geq 1}$ eventually stabilizes. This implies $Y$ is Noetherian.

(iI) It suffices to show that $A^n$ satisfies the ascending chain condition on open sets. If $\{U_i\}$ be an ascending chain of open sets, let $Y_i$ be the complement of $U_i$ in $A^n$. Then $\{Y_i\}$ is a descending chain of closed sets. Since taking ideal reverses inclusions, we conclude that $\mathcal{I}(Y_i)$ is an ascending chain of ideals in $k[X_1, \ldots, X_n]$. This chain should eventually stabilize, so

$$\mathcal{I}(Y_n) = \mathcal{I}(Y_{n+1}) = \ldots,$$

for $n$ large enough. Applying $Z$ to these equalities, we obtain

$$Y_n = Y_{n+1} = \ldots.$$

This clearly implies that the chain $\{U_i\}$ stabilizes.

(iii) Order the open sets in $X$ by inclusion. It is clear that any collection $\mathfrak{U}$ of open sets in $X$ must have a maximal element. Indeed, if this was false, we could inductively produce a strictly increasing chain of open sets, violating the fact that $X$ is Noetherian.

Now, consider an open cover

$$X = \bigcup_{\alpha} U_{\alpha},$$

and define the collection $\mathfrak{U}$ consisting in finite union of the open sets $U_{\alpha}$. The collection $\mathfrak{U}$ must have a maximal element $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. We claim

$$X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

Indeed, otherwise we could find

$$x \in X \setminus (U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}).$$

Now, $x \in U_{\beta}$ for some index $\beta$. Then,

$$U_{\beta} \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

belongs to the collection $\mathfrak{U}$, and strictly contains the maximal element of $\mathfrak{U}$, namely $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. This is absurd, thus proving our claim. \qed
Problem 3. Let $\mathbb{A}^3$ be the 3-dimensional affine space with coordinates $x, y, z$. Find the ideals of the following algebraic sets:

(i) The union of the $(x, y)$-plane with the $z$-axis.

(ii) The image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \to (t, t^2, t^3)$.

Answer: (i) The ideal of the $(x, y)$-plane is $(z)$. The ideal of the $z$-axis is $(x, y)$. By Problem 6, the ideal we’re looking for is $(z) \cap (x, y) = (xz, yz)$.

(ii) Let $I$ be the ideal of the set $(t, t^2, t^3)$ in $\mathbb{A}^3$, as $t \in \mathbb{A}^1$. Observe $(z - x^3, y - x^2)$ is clearly contained in $I$. We claim that $I = (z - x^3, y - x^2)$. We need to show every $f \in I$ is in the ideal $(z - x^3, y - x^2)$. Indeed, let us consider the image of $f$ in the quotient ring $\mathbb{A} = \mathbb{C}[x, y, z] / (z - x^3, y - x^2)$.

We would like to show that the image of $f$ is 0 in $\mathbb{A}$. Now, in the ring $\mathbb{A}$ the relations $z = x^3, y = x^2$ are satisfied. Thus

$$f(x, y, z) = f(x, x^2, x^3) \text{ in } \mathbb{A}.$$ 

Consider the polynomial $g(x) = f(x, x^2, x^3)$. Since $f(t, t^3, t^5) = 0$, we see $g(t) = 0$ for all $t$. Therefore $g \equiv 0$ as a polynomial in $x$. (This uses that the ground field is infinite). Therefore,

$$f(x, y, z) = 0 \text{ in } \mathbb{A} = \mathbb{C}[x, y, z] / (z - x^3, y - x^2),$$

which is what we wanted. 

□

Problem 4. Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map i.e. $f(p) = (f_1(p), \ldots, f_m(p))$ for $p \in \mathbb{A}^n$, where $f_1, \ldots, f_m$ are polynomials in $n$ variables. Are the following true or false:

(1) The image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ is an affine algebraic set.

(2) The inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.

(3) If $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph $\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic set.

Answer: (i) False. Consider the following counterexample which works over infinite ground fields. Consider the hyperbola

$$X = \{(x, y) : xy - 1 = 0\} \subset \mathbb{A}^2.$$
Clearly, $X$ is an algebraic set. Let $f : \mathbb{A}^2 \to \mathbb{A}^1$, $f(x, y) = x$
be the projection onto the first axis. The image of $f$ in $\mathbb{A}^1$ is $\{x : x \neq 0\}$. This
is not an algebraic set because the only algebraic subsets of $\mathbb{A}^1$ are empty set, finitely many points or $\mathbb{A}^1$.

(ii) True. Suppose $X = \mathcal{Z}(F_1, \ldots, F_s) \subseteq \mathbb{A}^m$, where $F_1, \ldots, F_s$ are polynomials in $m$ variables. Define

$$G_i = F_i(f_1(X_1, \ldots, X_n), \ldots, f_m(X_1, \ldots, X_n)), 1 \leq i \leq s.$$ 

These are clearly polynomials in $n$ variables. In short, we set

$$G_i = F_i \circ f, \ 1 \leq i \leq s$$

We claim $f^{-1}(X) = \mathcal{Z}(G_1, \ldots, G_s) \subseteq \mathbb{A}^n$,

which clearly exhibits $f^{-1}(X)$ as an algebraic set. This claim follows from the

remarks

$$p \in f^{-1}(X) \iff f(p) \in \mathcal{Z}(F_1, \ldots, F_s) \iff F_1(f(p)) = F_2(f(p)) = \ldots = F_s(f(p)) = 0$$

$$\iff G_1(p) = \ldots = G_s(p) \iff p \in \mathcal{Z}(G_1, \ldots, G_s).$$

(iii) True. Assume that $X$ is defined by the vanishing of the polynomials $F_1, \ldots, F_s$
in the variables $X_1, \ldots, X_n$. Set

$$G_1 = Y_1 - f_1(X_1, \ldots, X_n), \ldots, G_m = Y_m - f_m(X_1, \ldots, X_n).$$

We regard the $F$ and $G$'s as polynomials in the $n+m$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$.

We claim that the graph $\Gamma$ is defined by the equations

$$\Gamma = \mathcal{Z}(F_1, \ldots, F_s, G_1, \ldots, G_s) \subseteq \mathbb{A}^{n+m}.$$ 

Indeed,

$$(x, y) \in \mathcal{Z}(F_1, \ldots, F_s, G_1, \ldots, G_s) \iff F_1(x) = \ldots = F_s(x) = 0, y_1 = f_1(x), \ldots, y_m = f_m(x)$$

$$\iff x \in X, y = f(x) \iff (x, y) \in \Gamma.$$ 

\[\square\]

**Problem 5.** Let $X_1, X_2$ be affine algebraic sets in $\mathbb{A}^n$. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by an example that taking radicals in (ii) is necessary.
Answer: 
(i) If \( f \in I(X_1 \cup X_2) \), \( f \) vanishes on \( X_1 \cup X_2 \). In particular, \( f \) vanishes on \( X_1 \). Thus, \( f \in I(X_1) \). Similarly, we have \( f \in I(X_2) \). Then \( f \in I(X_1) \cap I(X_2) \). We proved that 
\[ I(X_1 \cup X_2) \subseteq I(X_1) \cap I(X_2). \]
On the other hand, if \( f \in I(X_1) \cap I(X_2) \), then \( f \in I(X_1) \) and \( f \in I(X_2) \). Therefore, \( f \) vanishes on both \( X_1 \) and \( X_2 \), and then also on \( X_1 \cup X_2 \). This implies that \( f \in I(X_1 \cup X_2) \). Hence 
\[ I(X_1) \cap I(X_2) \subseteq I(X_1 \cup X_2). \]
(ii) We assume the base field is algebraically closed. Because \( X_1, X_2 \) be affine algebraic sets, we can assume 
\[ X_1 = \mathcal{Z}(a), X_2 = \mathcal{Z}(b). \]
Furthermore, we may assume \( a \) and \( b \) are radical. Otherwise, if for instance \( a \) is not radical, we can replace \( a \) by \( \sqrt{a} \). This doesn’t change the algebraic sets in question as 
\[ X_1 = \mathcal{Z}(a) = \mathcal{Z}(\sqrt{a}). \]
By Hilbert’s Nullstellensatz, 
\[ \sqrt{I(X_1) + I(X_2)} = \sqrt{I(\mathcal{Z}(a)) + I(\mathcal{Z}(b))} = \sqrt{\sqrt{a} + \sqrt{b}} = \sqrt{a + b} \]
and 
\[ I(X_1 \cap X_2) = I(\mathcal{Z}(a) \cap \mathcal{Z}(b)) = I(\mathcal{Z}(a + b)) = \sqrt{a + b}. \]
Therefore, 
\[ I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}. \]
Note: In the above, we made use of the equality 
\[ \mathcal{Z}(a) \cap \mathcal{Z}(b) = \mathcal{Z}(a + b). \]
This can be argued as follows. If \( x \in \mathcal{Z}(a) \cap \mathcal{Z}(b) \), then \( f(x) = g(x) = 0 \) for all \( f \in a \) and \( g \in b \). Thus \( x \in \mathcal{Z}(a + b) \) proving that 
\[ \mathcal{Z}(a) \cap \mathcal{Z}(b) \subseteq \mathcal{Z}(a + b). \]
Conversely, if \( x \in \mathcal{Z}(a + b) \), then \( (f + g)(x) = 0 \) for every \( f \in a \) and \( g \in b \). In particular, picking \( g = 0 \), we get \( f(x) = 0 \) for every \( f \in a \), so \( x \in \mathcal{Z}(a) \). Similarly \( x \in \mathcal{Z}(b) \). Therefore 
\[ \mathcal{Z}(a + b) \subseteq \mathcal{Z}(a) \cap \mathcal{Z}(b). \]
Counterexample: Consider the following two algebraic subsets of \( \mathbb{A}^2 \):
\[ X_1 = (y - x^2 = 0), X_2 = (y = 0). \]
Graphically, these can be represented by a parabola and a line tangent to it. It is clear that

\[ X_1 \cap X_2 = \{0\}, \text{ so } I(X_1 \cap X_2) = (x, y). \]

On the other hand,

\[ I(X_1) + I(X_2) = (y - x^2) + (y) = (y - x^2, y) = (y, x^2). \]

Therefore

\[ I(X_1 \cap X_2) \neq I(X_1) + I(X_2). \]

\[ \square \]

**Problem 6.** Find the irreducible components of the affine algebraic set \(xz - y^2 = z^3 - x^5 = 0\) in \(\mathbb{A}^3\).

**Answer:** If \(x = 0\), then the two equations imply that \(y = z = 0\). Similarly, if \(y = 0\), then \(x = z = 0\). Let us assume that \(x \neq 0\), \(y \neq 0\).

Write the first equation as

\[ \frac{y}{x} = \frac{z}{y} := t. \]

Thus,

\[ y = tx, \ z = t^2x. \]

The second equation becomes

\[ z^3 = x^5 \iff t^6 x^3 = x^5 \iff x^2 = t^6 \iff x = \pm t^3. \]

This implies

\[ y = \pm t^4, \ z = \pm t^5. \]

Thus \((x, y, z) = (t^3, t^4, t^5)\) or \((-t^3, -t^4, -t^5)\) for some \(t \in \mathbb{C}\).

Letting

\[ X_1 = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\} \quad \text{and} \quad X_2 = \{(-t^3, -t^4, -t^5) \mid t \in \mathbb{C}\} \]

we obtain

\[ X = X_1 \cup X_2. \]

We claim that \(X_1, X_2\) are the irreducible components of \(X\).

First, there are polynomial maps

\[ f_1 : \mathbb{A}^1 \to \mathbb{A}^3 \quad t \to (t^3, t^4, t^5) \quad \text{and} \quad f_2 : \mathbb{A}^1 \to \mathbb{A}^3 \quad t \to (-t^3, -t^4, -t^5). \]

Since \(X_1 = f_1(\mathbb{A}^1)\), \(X_2 = f_2(\mathbb{A}^1)\) and \(\mathbb{A}^1\) is irreducible, the images \(X_1\) and \(X_2\) are irreducible.

It remains to explain that \(X_1\) and \(X_2\) are algebraic sets. We claim

\[ X_1 = \mathcal{Z}(y^3 - x^4, z^3 - x^5, z^4 - y^5), \quad X_2 = \mathcal{Z}(y^3 + x^4, z^3 - x^5, z^4 + y^5). \]
Let us check this claim for $X_1$ only. This follows using the same strategy as before. Indeed, if $x = 0$ then $y = z = 0$, so we may assume $x \neq 0$. Let 
$$t = \frac{y}{x},$$
From the first equation we have 
$$x = \left(\frac{y}{x}\right)^3 = t^3 \implies y = t^4.$$ 
Dividing the last two equations we get 
$$z = \frac{y^5}{x^5} = t^5 \implies (x, y, z) = (t^3, t^4, t^5),$$
as claimed. The verification for $X_2$ is entirely similar.

\[ \square \]

**Problem 7.** Let $Y$ be a subspace of a topological space $X$. Show that $Y$ is irreducible if and only if the closure of $Y$ in $X$ is irreducible.

**Answer:** Assume $Y$ is irreducible. If $\overline{Y} = Y_1 \cup Y_2$, with $Y_1, Y_2$ are proper closed subsets in $\overline{Y}$. Then $Y_1, Y_2$ are closed in $X$. Note that 
$$Y = (Y_1 \cap Y) \cup (Y_2 \cap Y),$$
and $Y_1 \cap Y, Y_2 \cap Y$ are closed in $Y$. This implies that for some $1 \leq i \leq 2$, 
$$Y_i \cap Y = Y \implies Y \subset Y_i \implies \overline{Y} \subset Y_i (= \text{closed})$$
which is a contradiction.

The converse is similar. Assume $\overline{Y}$ is irreducible, and that $Y = Y_1 \cup Y_2$, with $Y_1, Y_2$ closed in $Y$. Write
$$Y_i = F_i \cap Y,$$ 
where $F_1, F_2$ are closed in $X$. We have 
$$Y \subset F_1 \cup F_2 \implies \overline{Y} \subset F_1 \cup F_2 \implies \overline{Y} = (F_1 \cap \overline{Y}) \cup (F_2 \cap \overline{Y}).$$
Hence, for some $i$, 
$$F_i \cap \overline{Y} = \overline{Y} \implies \overline{Y} \subset F_i \implies Y \subset F_i \cap Y = Y_i \implies Y_i = Y.$$

\[ \square \]

**Problem 8.** Let $X$ be the union of the three coordinate axes in $\mathbb{A}^3$. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements.

**Answer:** We claim that 
$$I(X) = (xy, yz, zx).$$
Clearly, $xy, yz, zx$ vanish on $X$. Conversely, if a polynomial
\[ f = \sum_{(i,j,k)} a_{ikj} x^i y^j z^k \]
vanishes on the $x$-axis, then $f(x, 0, 0) = 0$, hence
\[ a_{i00} = 0, \text{ for } i \geq 0. \]

Similarly,
\[ a_{i00} = a_{0j0} = a_{00k} = 0, \text{ for all } i, j, k \geq 0. \]

This means that
\[ f \in \langle xy, yz, zx \rangle. \]

Now, $I(X)$ cannot be generated by one polynomial $f$ since then $f$ will have to divide $xy, yz, zx$ which have no common factors.

Assume that $I(X)$ is generated by two elements $(f, g)$. Since $f \in \langle xy, yz, zx \rangle$, we see that
\[ f = xyP + yzQ + zxR \]
for some polynomials $P, Q, R$. Letting $\tilde{f}$ be the degree 2 component of $f$, and $p, q, r$ be the free terms in $P, Q, R$, we obtain that
\[ \tilde{f} = pxy + qyz + rzx \]
only contains the monomials $xy, yz, zx$. The similar remark applies to the degree 2 piece $\tilde{g}$ of the polynomial $g$.

Furthermore, there must be polynomials $A_1, B_1$ such that
\[ yz = A_1 f + B_1 g. \]

Look at the degree 2 terms of the above equality. Letting $a_1, b_1$ be the free terms in $A_1, B_1$, we obtain
\[ yz = a_1 \tilde{f} + b_1 \tilde{g}. \]

Similarly,
\[ xz = a_2 \tilde{f} + b_2 \tilde{g}, \]
\[ xy = a_3 \tilde{f} + b_3 \tilde{g}. \]

To reach a contradiction, consider the $V$ the three dimensional vector space of polynomials of degree 2 which contain the monomials $xy, yz, zx$ only. We observed that
\[ \tilde{f}, \tilde{g} \in V. \]

Moreover, the last three equations above show that $\tilde{f}$ and $\tilde{g}$ span the three dimensional vector space $V$. This is a contradiction showing that $I(X)$ cannot be generated by fewer than 3 elements. \[ \square \]