Math 203A, Solution Set 2.

Problem 1. An algebraic set \( C \subset A^2 \) defined by an irreducible polynomial \( F \) of degree 2 is called an irreducible conic.

(i) Show that any irreducible conic is isomorphic to
\[
Y - X^2 = 0 \quad \text{or} \quad XY - 1 = 0.
\]

(ii) Let \( C_1 \) and \( C_2 \) be two distinct irreducible conics in \( A^2 \). Using (i), show that \( C_1 \) and \( C_2 \) intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

Answer: Let
\[
F(x, y) = ax^2 + by^2 + cxy + dx + ey + f.
\]
We will show that after a linear/affine change of coordinates the conic can be written as
\[
XY - 1 = 0 \quad \text{or} \quad Y - X^2 = 0.
\]

We discuss the following cases.

Case 1. If \( a = b = 0 \), then
\[
F(x, y) = cxy + dx + ey + f = c\left(xy + \frac{d}{c}x + \frac{e}{c}y\right) + f = c\left(x + \frac{c}{e}\right)\left(y + \frac{d}{c}\right) + \tilde{f}
\]
If \( \tilde{f} = 0 \), \( F \) is reducible, which is not allowed. Therefore \( \tilde{f} \neq 0 \). Let
\[
X = -\frac{c}{\tilde{f}}\left(x + \frac{c}{e}\right) \quad \text{and} \quad Y = y + \frac{d}{c},
\]
so that
\[
F = -\tilde{f}XY + \tilde{f}.
\]
Thus \( F(x, y) = 0 \) implies
\[
XY - 1 = 0.
\]
After an affine change of coordinates, the conic \( Z(F) \) becomes
\[
XY - 1 = 0.
\]

Case 2. If either \( a \) or \( b \) is not 0, without loss of generality, we may assume \( a \neq 0 \). Then,
\[
F = ax^2 + by^2 + cxy + dx + ey + f = a\left(x + \frac{c}{2a}y\right)^2 + \tilde{b}y^2 + dx + ey + f
\]
Let
\[
x_1 = \sqrt{a}\left(x + \frac{c}{2a}y\right) \quad \text{and} \quad y_1 = y,
\]
(choose any one of the square roots). There exist constants \( \tilde{d} \), \( \tilde{e} \), \( \tilde{f} \) such that
\[
F = x_1^2 + \tilde{b}y_1^2 + \tilde{d}x_1 + \tilde{e}y_1 + \tilde{f} = \left(x_1 + \frac{\tilde{d}}{2}\right)^2 + \tilde{b}y_1^2 + \tilde{e}y_1 + \tilde{f}.
\]
Let

\[ x_2 = x_1 + \frac{\tilde{d}}{2}. \]

Subcase (i): If \( b = 0 \),

\[ F = x_2^2 + \tilde{e}y_1 + \tilde{\tilde{f}}. \]

We claim \( \tilde{e} \neq 0 \) because otherwise

\[ F = (x_2 + i\sqrt{\tilde{f}})(x_2 - i\sqrt{\tilde{f}}) \]

is reducible. Let

\[ X = x_2, \ Y = -(\tilde{e}y_1 + \tilde{\tilde{f}}). \]

Then

\[ F = X^2 - Y. \]

Subcase (ii): If \( b \neq 0 \), let

\[ y_2 = \sqrt{b}y_1 + \frac{\tilde{e}}{2\sqrt{b}} \]

so that

\[ F = x_2^2 + y_2^2 + g. \]

Letting

\[ X = \sqrt{-g}(x_2 + iy_2) \quad \text{and} \quad Y = \sqrt{-g}(x_2 - iy_2), \]

we have

\[ F = -g(XY - 1). \]

Therefore, the conic \( Z(F) \) can be written in the form

\[ XY - 1 = 0 \]

after an affine change of coordinates.

(ii) Suppose \( C_1 = Z(f) \) and \( C_2 = Z(g) \) for some polynomials \( f, g \in \mathbb{C}[X,Y] \). Using (i), we can change coordinates to achieve

\[ f = XY - 1 \] or \( f = Y - X^2. \]

Let us write

\[ g = aX^2 + bY^2 + cXY + dX + eY + f. \]

Let \( (x, y) \in C_1 \cap C_2 \), i.e.

\[ f(x, y) = g(x, y) = 0. \]
Case 1. If \( f = XY - 1 \), then
\[
y = \frac{1}{x} \text{ and } x \neq 0.
\]
Substituting into \( g \), we get
\[
0 = g(x, 1/x) = ax^2 + \frac{b}{x^2} + cx \cdot \frac{1}{x} + dx + e + f.
\]
Multiplying by \( x^2 \), we obtain
\[
ax^4 + b + cx^2 + dx^3 + ex + fx^2 = 0.
\]
This degree 4 polynomial has at most 4 roots. Therefore, \( C_1 \cap C_2 \) contains at most 4 points.

Case 2. If \( f = Y - X^2 \), then
\[
y = x^2.
\]
Substituting into \( g \), we get
\[
0 = g(x, x^2) = ax^2 + bx^4 + cx^3 + dx + ex^2 + f.
\]
Thus \( x \) satisfies a polynomial of degree at most 4, so there are at most 4 values for \( x \). As \( y = x^2 \), it follows that \( C_1 \cap C_2 \) contains at most 4 points.

The following examples show that the number of intersection points can be either 0, 1, 2, 3, 4:

(i) The conics \( XY - 1 = 0 \) and \( X^2 + XY - 1 = 0 \) intersect in 0 points.
(ii) The conics \( Y - X^2 = 0 \) and \( Y^2 + Y - X^2 = 0 \) intersect in 1 point.
(iii) The conics \( Y - X^2 = 0 \) and \( Y^2 + XY + Y - X^2 = 0 \) intersect in 2 points.
(iv) The conics \( XY - 1 = 0 \) and \( Y - X^2 = 0 \) intersect in 3 points.
(v) The conics \( X - Y^2 = 0 \) and \( Y - X^2 = 0 \) intersect in 4 points.

\[\square\]

Problem 2. Which of the following algebraic sets are isomorphic:

(i) \( \mathbb{A}^1 \)
(ii) \( \mathbb{Z}(xy) \subset \mathbb{A}^2 \)
(iii) \( \mathbb{Z}(x^2 + y^2) \subset \mathbb{A}^2 \)
(iv) \( \mathbb{Z}(x^2 - y^5) \subset \mathbb{A}^2 \)
(v) \( \mathbb{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2 \).

Answer: We claim that (i) and (v) are isomorphic, (ii) and (iii) are isomorphic, and (iv) is not isomorphic to any other algebraic sets.

- We check (i) and (v) are isomorphic. Note first that
\[
\mathbb{Z}(y - x^2, z - x^3) = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}.
\]
Define the morphisms
\[ f : \mathbb{A}^1 \to \mathcal{Z}(y - x^2, z - x^3), \quad f(t) = (t, t^2, t^3), \]
and
\[ g : \mathcal{Z}(y - x^2, z - x^3) \to \mathbb{A}^1, \quad g(x, y, z) = x. \]
Then
\[ f \circ g = \text{identity} \quad g \circ f = \text{identity}. \]
Thus \( f \) and \( g \) are inverse isomorphisms.

- We show (ii) and (iii) are isomorphic. Consider the morphisms
  \[ f : \mathbb{A}^2 \to \mathbb{A}^2, \quad f(x, y) = (x + iy, x - iy), \]
  \[ g : \mathbb{A}^2 \to \mathbb{A}^2, \quad g(x, y) = \left(\frac{x + y}{2}, \frac{x - y}{2i}\right). \]
  It is easy to check that
  \[ f \circ g = \text{identity} \quad g \circ f = \text{identity}. \]
  Moreover, letting
  \[ Z = \mathcal{Z}(x^2 + y^2) \quad \text{and} \quad W = \mathcal{Z}(xy), \]
  we claim that
  \[ f(Z) \subset W, \quad g(W) \subset Z. \]
  Indeed, if \((x, y) \in Z\), then
  \[ f(x, y) = (x - iy, x + iy) \in W \quad \text{since} \quad (x - iy)(x + iy) = x^2 + y^2 = 0. \]
  Similarly, if \((x, y) \in W\), then
  \[ g(x, y) = \left(\frac{x + y}{2}, \frac{x - y}{2i}\right) \in Z, \quad \text{since} \quad \left(\frac{x + y}{2}\right)^2 + \left(\frac{x - y}{2i}\right)^2 = 0. \]
  Therefore, \( f \) and \( g \) establish isomorphisms between \( Z \) and \( W \). (Here we assume \( k \) is not of characteristic 2. If so, this set is isomorphic to \( \mathbb{A}^1 \).)

- We show that (i) and (ii) are not isomorphic. Indeed, \( \mathcal{Z}(xy) \) is the union of two lines, hence it is a reducible affine set. On the other hand, \( \mathbb{A}^1 \) is irreducible, hence it cannot be isomorphic to \( \mathcal{Z}(xy) \).

- We show (ii) and (iv) cannot be isomorphic. This follows by the same argument observing that \( \mathcal{Z}(x^2 - y^5) \) is irreducible. Indeed, it suffices to prove that the polynomial \( x^2 - y^5 \) is irreducible. Assuming the contrary, write
  \[ x^2 - y^5 = f_1(x, y)f_2(x, y). \]
  Regarding \( f_1, f_2 \) as polynomials in \( x \) with coefficients in the integral domain \( k[y] \), we conclude that \( f_1, f_2 \) can have degree at most 2 with respect to \( x \). In fact, it
is clear that the combination of degrees \((0,2)\) cannot occur. If the degrees are 1, we may assume

\[
f_1(x, y) = x - g(y), \quad f_2(x, y) = x - h(y).
\]

Then

\[
x^2 - y^5 = x^2 - x(g(y) + h(y)) + g(y)h(y).
\]

Therefore,

\[
g(y) = -h(y), \quad \text{and} \quad -y^5 = g(y)h(y) = -g(y)^2.
\]

This is clearly impossible, proving our claim.

- We show that (i) and (iv) cannot be isomorphic. Letting \(t\) be the coordinate of \(\mathbb{A}^1\), we show that there cannot be an isomorphism

\[
\Phi : k[x, y]/(x^2 - y^5) \to k[t].
\]

Indeed, set

\[
\Phi(x) = p, \quad \Phi(y) = q.
\]

We must have

\[
p^2 = q^5.
\]

This implies that all irreducible factors appearing in \(q\) have even exponent, so

\[
q = r^2, \quad p = r^5
\]

for some polynomial \(r\). Note that \(r\) cannot be constant since otherwise the image of \(\Phi\) would have to consist in constant polynomials.

Now since \(\Phi\) is surjective, there is a polynomial

\[
f = \sum_{i,j} a_{ij}x^iy^j
\]

such that \(\Phi(f) = r\). This means that

\[
\sum_{ij} a_{ij}r^{5i+2j} = r.
\]

In particular, since the left hand side must be divisible by \(r\), we have \(a_{00} = 0\). However, since \(5i + 2j \geq 2\) for \((i,j) \neq (0,0)\), the left hand side is in fact divisible by \(r^2\), so it cannot equal \(r\). This contradiction shows that an isomorphism \(\Phi\) cannot exist.

\(\square\)

**Problem 3.** Show that \(X = \mathbb{A}^2 \setminus \{(0,0)\}\) cannot be isomorphic to an affine algebraic set.
Answer: Assume otherwise, and let \( \Phi : Y \to X \) be an isomorphism between an affine algebraic set \( Y \) and \( X \). Consider the composition \( \Psi = \iota \circ \Phi \), where \( \iota : X \to \mathbb{A}^2 \) is the inclusion. Note that \( g \) is a regular function on \( Y \) if and only if \( g \circ \Phi^{-1} \) is a regular function on \( X \) if and only if \( g \circ \Phi^{-1} \in k[x, y] \) by the removable singularity theorem. Letting \( P = g \circ \Phi^{-1} \), we have
\[
g = P \circ \Psi
\]
for some polynomial \( P \in k[x, y] \). Thus, \( \Psi^* \) induces an isomorphism between the coordinate ring of \( Y \) and \( k[x, y] \) proving that \( \Psi : Y \sim A^2 \) is an isomorphism. In turn, this means \( \iota : X \sim A^2 \) is an isomorphism. This is impossible because \( \iota : X \to A^2 \) is not bijective.

\[\square\]

Problem 4. Let \( n \geq 2 \), and let \( S = \{a_1, \ldots, a_n\} \) be a finite set with \( n \) elements in \( \mathbb{A}^1 \).

(i) Show that the quasi-affine set \( \mathbb{A}^1 \setminus S \) is isomorphic to an affine set. For instance, you may take \( X \) to be the affine algebraic set given by the equations
\[
X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.
\]
Show that the projection onto the first coordinate
\[
\pi : X \to \mathbb{A}^1 \setminus S, \ (X_0, \ldots, X_n) \mapsto X_0
\]
is an isomorphism.

(ii) Show that \( \mathbb{A}^1 \setminus S \) is not isomorphic to \( \mathbb{A}^1 \setminus \{0\} \) by proving that their rings of regular functions are not isomorphic.

Answer: (i) If \( (X_0, \ldots, X_n) \in X \) we have \( X_i(X_0 - a_i) = 1 \) hence \( X_0 \neq a_i \) for all \( 1 \leq i \leq n \). Therefore
\[
\pi : X \to \mathbb{A}^1 \setminus S
\]
is well defined. Let
\[
\phi : \mathbb{A}^1 \setminus S \to X, t \mapsto \left( t, \frac{1}{t-a_1}, \ldots, \frac{1}{t-a_n} \right).
\]
Both \( \pi \) and \( \phi \) are rational maps, regular everywhere. It is clear that
\[
\pi \circ \phi = \text{identity and } \phi \circ \pi = \text{identity}.
\]
Therefore, \( \pi \) and \( \phi \) are inverse isomorphisms.

(ii) Assume there is an isomorphism
\[
\Phi : A(X) \to k[t, t^{-1}].
\]
Since
\[
X_i(X_0 - a_i) = 1 \text{ in } A(X),
\]
it follows that
\[ \Phi(X_i) \Phi(X_0 - a_i) = 1. \]
Writing
\[ \Phi(X_i) = \frac{g_i(t)}{t^{\alpha_i}}, \quad \Phi(X_0 - a_i) = \frac{h_i(t)}{t^{\beta_i}}, \]
for some polynomials \( g_i, h_i \), we obtain
\[ g_i(t)h_i(t) = t^{\alpha_i + \beta_i}. \]
This implies that \( h_i \) is of the form \( ct^m \), or equivalently
\[ \Phi(X_0 - a_i) = c_i t^{m_i} \]
for some \( m_i \in \mathbb{Z} \) and \( c_i \in k \). Subtracting the relations for \( i \) and \( j \), it follows that
\[ a_j - a_i = \Phi(a_j - a_i) = c_j t^{m_j} - c_i t^{m_i}. \]
Comparing degrees, we see that this implies \( m_i = m_j = 0 \), as \( a_i \neq a_j \) for \( i \neq j \).
In turn, we obtain
\[ \Phi(X_0 - a_i) = c_i. \]
Since \( \Phi(c_i) = c_i \), we contradicted the injectivity of \( \Phi \). This shows that \( \Phi \) cannot be an isomorphism completing the proof.

\[ \square \]

**Problem 5.** Let \( n \geq 2 \). Consider the affine algebraic sets in \( \mathbb{A}^2 \):
\[ Z_n = \mathbb{Z}(y^n - x^{n+1}) \]
and
\[ W_n = \mathbb{Z}(y^n - x^n(x + 1)). \]
Show that \( Z_n \) and \( W_n \) are birational but not isomorphic.

(i) Show that
\[ f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1}) \]
is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets
\[ \mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}. \]
Similarly, show that
\[ g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t) \]
is a morphism of affine algebraic sets. Find open subsets of \( \mathbb{A}^1 \) and \( W_n \) where \( g \) becomes an isomorphism.

(ii) Using (i), explain why \( Z_n \) and \( W_n \) are birational. Write down a birational isomorphism between \( Z_n \to W_n \).
(iii) Assume that there exists an isomorphism

\[ h : \mathbb{Z}_n \rightarrow W_n \]

such that \( h((0,0)) = (0,0) \). Observe that this induces an isomorphism between the open sets

\[ Z_n \setminus \{(0,0)\} \rightarrow W_n \setminus \{(0,0)\}. \]

Use part (i) and the previous problem to conclude this cannot be true if \( n \geq 2 \).

(iv) Repeat the argument above without the assumption that \( h \) sends the origin to itself. You may need to prove a stronger version of Problem 4.

Answer:

(i) It is clear that

\[ f : \mathbb{A}^1 \rightarrow \mathbb{Z}_n, \ t \rightarrow (t^n, t^{n+1}) \]

is a well defined morphism. Consider the morphism

\[ f^{-1} : Z_n \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus \{0\} \]

given by

\[ (x, y) \mapsto \frac{y}{x}. \]

A direct computation shows that \( f^{-1} \) is the inverse morphism of \( f \). Similarly,

\[ g : \mathbb{A}^1 \rightarrow W_n, \ t \rightarrow (t^n - 1, t^{n+1} - t) \]

is a well defined morphism. Its inverse morphism is

\[ g^{-1} : W_n \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus S, \ (x, y) \mapsto \frac{y}{x}. \]

Here, \( S \) is the set of all \( n \) roots of unity. The two morphisms \( g \) and \( g^{-1} \) establish an isomorphism between

\[ \mathbb{A}^1 \setminus S \rightarrow W_n \setminus \{0\}. \]

(ii) Part (i) shows that both \( \mathbb{Z}_n \) and \( W_n \) are birational to \( \mathbb{A}^1 \) so they are birational to each other. An explicit birational isomorphism is

\[ g \circ f^{-1} : Z_n \rightarrow W_n. \]

A direct computation shows

\[ g \circ f^{-1}(x, y) = \left( \frac{y^n}{x^n}, \frac{y^{n+1}}{x^{n+1}} - \frac{y}{x} \right). \]

(iii) If \( h : Z_n \rightarrow W_n \) is an isomorphism sending the origin to itself, then

\[ h : Z_n \setminus \{0\} \rightarrow W_n \setminus \{0\} \]

is also an isomorphism. By part (i), \( g^{-1} \circ h \circ f \) induces an isomorphism between the quasi-affine sets

\[ \mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus S. \]

Such an isomorphism cannot exist by Problem 1.
Problem 6. Let $X$ be an affine variety, and let $G$ be a finite group. Assume that $G$ acts on $X$ algebraically, i.e. that for every $g \in G$, we are given a morphism $g : X \to X$ (denoted by the same letter for simplicity of notation), such that 

$$(gh)(p) = g(h(p))$$

for all $g, h \in G$ and $p \in X$.

(i) Let $g \in G$ act on the coordinate rings $A(X)$ via

$$f \mapsto f^g \text{ with } f^g(p) = f(g(p)).$$

Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all $G$-invariant functions on $X$. Show that $A(X)^G$ is a finitely generated $k$-algebra.

(ii) By (i), there is an affine variety $Y$ with coordinate ring $A(X)^G$, together with a morphism

$$\pi : X \to Y$$

determined by the inclusion

$$A(X)^G \hookrightarrow A(X).$$

Show that $Y$ can be considered as the quotient of $X$ by $G$, denoted $X/G$, in the following sense: if $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(iii) Let

$$\mu_n = \left\{ \exp \left( \frac{2\pi ik}{n} \right) : k \in \mathbb{Z} \right\}$$

be the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Describe $\mathbb{C}/\mu_n$ and $\mathbb{C}^2/\mu_n$ as affine algebraic sets.

Answer: (i) Write $|G| = m$. If $R = A(X) = k[x_1, \ldots, x_n]/I$ is a finitely generated $k$-algebra, then we claim that $R^G$ is also finitely generated $k$-algebra. Indeed, for all $1 \leq i \leq n$, consider the polynomial

$$F_i(t) = \prod_{g \in G} (t - x_i^g) \in R[t].$$

This polynomial has coefficients $f_{ij}$ which are the $j^{\text{th}}$ elementary symmetric functions in $x_i^g$. Clearly, $f_{ij} \in R^G.$ Note furthermore that

$$F_i(x_i) = 0 \implies x_i^m = - \sum_{j<m} f_{ij} x_i^j.$$

Consider $S$ the subalgebra of $R^G$ generated by $f_{ij}$. Thus $x_i^m$ can be expressed as a combination of $1, \ldots, x_i^{m-1}$ with coefficients in $S$. Therefore, all monomials

$$x_1^{m_1} \cdots x_n^{m_n}$$
can be expressed in terms of the monomials with $m_i < m$, and coefficients in $S$. Going further, any $f \in R$ can be written as

$$f = \sum_I s_I x^I$$

where $s_I \in S$ and $I = (m_1, \ldots, m_n)$ has $m_i < m$. Let

$$\pi(a) = \frac{1}{m} \sum_{g \in G} a^g$$

denote the averaging operators. Then if $f \in f^G$ we have

$$f = \pi(f) = \sum_I \pi(s_I x^I) = \sum_I s_I \pi(x^I),$$

using that $s_I \in S \subset R^G$ is invariant by the group $G$. In consequence, $R^G$ is generated by $\pi(x^I)$ for finitely many multi-indices $I$. Thus $R^G$ is a finitely generated $k$-algebra.

(ii) Assume that $q = g \cdot p$. We show that if $f$ is a regular function on $Y$ then

$$f(\pi(p)) = f(\pi(q)).$$

This will be true about the coordinate functions in particular, showing that

$$\pi(p) = \pi(q)$$

as claimed. Indeed, if $f$ is regular on $Y$, then by definition it corresponds to a $G$-invariant function on $X$, which we also denote by $\tilde{f}$. We have $\tilde{f} = f \circ \pi$. Since $q = g \cdot p$ and $\tilde{f}$ is $G$-invariant, then

$$\tilde{f}(p) = \tilde{f}(q) \implies f(\pi(p)) = f(\pi(q)).$$

Conversely, assume $\pi(p) = \pi(q)$. Then by the same argument, all $G$-invariant function $\tilde{f}$ have

$$\tilde{f}(p) = \tilde{f}(q).$$

Let $W_1$ and $W_2$ be the $G$-orbits of $p, q$. We need to show that the orbits coincide, so let us assume they do not, so that in this case $W_1, W_2$ are both closed disjoint sets. Thus $Z(I(W_1) + I(W_2)) = W_1 \cap W_2 = \emptyset$ by problem 6(ii) in the first problem set. By the Nullstellensatz,

$$1 \in I(W_1) + I(W_2)$$

so we can write

$$1 = f_1 + f_2$$

with $f_1, f_2$ vanishing on $W_1, W_2$. In particular $f = 1$ on $W_2$. Replacing $f_1$ by its average

$$\tilde{f} = \frac{1}{|G|}(\sum_{h \in G} f_1^h),$$
we obtain a $G$-invariant regular function such that $\tilde{f} = 0$ on $W_1$ and 1 on $W_2$. This is what we wanted to prove.

(iii) The coordinate ring of $\mathbb{C}/\mu_n$ is $\mathbb{C}[z]^{\mu_n} = \mathbb{C}[z^n]$. Writing $w = z^n$, it is clear then that $\mathbb{C}/\mu_n$ is isomorphic to $\mathbb{C}$. The coordinate ring of $\mathbb{C}^2/\mu_n$ is $\mathbb{C}[z,w]^{\mu_n}$. The invariant polynomials are

$$z^n, z^{n-1}w, \ldots, zw^{n-1}, w^n.$$

Writing $x_i = z^{n-i}w^i$ in $\mathbb{A}^{n+1}$, we obtain the equations

$$x_1x_3 = x_2^2, \ x_2x_4 = x_3^2, \ldots, \ x_{n-1}x_{n+1} = x_n^2.$$