Problem 1. Let $V$ be a finite dimensional vector space and let $\omega \in \Lambda^2 V$ be such that $\omega \wedge \omega = 0$. Show that $\omega = v \wedge w$ for some vectors $v, w \in V$.

Answer: It is clear that if $\omega = v \wedge w$ then $\omega \wedge \omega = 0$. Conversely, we will induct on $n$, the base case $n = 2$ being clear. Let us write

$$\omega = e_0 \wedge \eta + \omega'$$

where $\omega', \eta$ do not contain the vector $e_0$. Thus

$$0 = \omega \wedge \omega = 2e_0 \wedge \eta \wedge \omega' + \omega' \wedge \omega'.$$

This implies that

$$\omega' \wedge \omega' = 0$$

hence by induction

$$\omega' = v \wedge w,$$

with $v, w$ being in the subspace spanned by $e_1, \ldots, e_n$. Also, we know

$$e_0 \wedge \eta \wedge \omega' = 0 \implies \eta \wedge v \wedge w = 0.$$ 

This shows that $\eta$ cannot be independent of $v, w$ hence

$$\eta = av + bw.$$ 

Collecting terms we find

$$\omega = e_0 \wedge (av + bw) + v \wedge w = (v + be_0) \wedge (w + ae_0)$$

as claimed.

\[\square\]

Problem 2. The set $X$ of degree $d$ homogeneous polynomials in $n+1$ variables can be identified with a projective space $\mathbb{P}^N$, by recording the coefficients in some order. What is $N$?

Show that the set of reducible polynomials form a closed subset of $X$.

Answer: The space $V_d$ of degree $d$ polynomials in $n+1$ variables has dimension $\binom{n+d}{d}$. Consider the morphism

$$\phi_k : \mathbb{P}(V_k) \times \mathbb{P}(V_{d-k}) \to \mathbb{P}(V_d), \quad (f,g) \mapsto f \cdot g.$$ 

Clearly, $\phi_k$ is a morphism. This can be seen by writing

$$f = \sum a_I z^I, \quad g = \sum b_J z^J \implies f \cdot g = \sum_K \left( \sum_{I+J=K} a_I b_J \right) z^K$$

which shows

$$\phi_k(a_I, b_J) = (c_K), \quad c_K = \sum_{I+J=K} a_I b_J.$$ 

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This is a morphism. By the main theorem of projective geometry,

\[ Y_k = \text{Image } \phi_k \]

is closed. The reducible polynomials are given as

\[ Y = \bigcup_{k=1}^{d-1} Y_k. \]

This set is therefore also closed in \( X \).

**Problem 3.** Show that \( \mathbb{P}^1 \times \mathbb{A}^1 \) and \( \mathbb{P}^2 \setminus \{x\} \) are neither affine nor projective.

**Answer:** Write \( t \) for the coordinate on \( \mathbb{A}^1 \). We claim that the regular functions on \( \mathbb{P}^1 \times \mathbb{A}^1 \) are polynomials \( f(t) \). This will show that \( \mathbb{P}^1 \times \mathbb{A}^1 \) is not projective, because projective varieties only have constants as regular functions. It also shows \( X = \mathbb{P}^1 \times \mathbb{A}^1 \) is not affine since it were affine, its coordinate ring would be

\[ A(X) = k[t] = A(\mathbb{A}^1). \]

Then \( X \simeq \mathbb{A}^1 \), but this is clearly impossible for dimension reasons.

Indeed, let \( U, V \) be two affine opens covering \( \mathbb{P}^1 \). We have

\[ U \simeq \mathbb{A}^1, \quad V \simeq \mathbb{A}^1 \]

with coordinates \( z, w \) and \( w = \frac{1}{z} \) over overlaps. Let \( \phi \) be a regular function on \( \mathbb{P}^1 \times \mathbb{A}^1 \). Then \( \phi \) is regular on \( U \times \mathbb{A}^1 \simeq \mathbb{A}^2 \) so

\[ \phi = p(z, t) \]

for some polynomial \( p \). Similarly, \( \phi \) is regular on \( V \times \mathbb{A}^1 \simeq \mathbb{A}^2 \) so

\[ \phi = q(w, t) \]

for some polynomial \( q \). Over \( (U \cap V) \times \mathbb{A}^1 \) we must have

\[ p(z, t) = q\left(\frac{1}{z}, t\right). \]

The powers of \( z \) on the left have nonnegative exponents, while the powers of \( z \) on the right have nonpositive exponents, so the exponents must be 0. Thus,

\[ p(z, t) = q\left(\frac{1}{z}, t\right) = f(t) \implies \phi = f(t) \]

for some polynomial \( f \).

For \( Y = \mathbb{P}^2 \setminus \{x\} \), we claim that all regular functions are constant. This will show that \( Y \) cannot be affine because

\[ A(Y) = k = A(\text{point}) \implies Y \simeq \text{point} \]
which is clearly impossible for dimension reasons. Indeed, if \( \phi \) is regular on \( \mathbb{P}^2 \setminus \{x\} \), then consider the restriction of \( \phi \) to \( U = \mathbb{A}^2 \setminus \{0\} \). This extends to a regular function on \( \mathbb{A}^2 \) by the removable singularity theorem. Thus \( \phi \) extends to a regular function on \( \mathbb{P}^2 \), showing then that \( \phi \) must be constant.

To see \( Y \) is not projective, assume otherwise. Let \( L = \{ \ell = 0 \} \) be a line in \( \mathbb{P}^2 \) through \( x \). Then, \( Z = L \setminus \{x\} \) is closed in \( Y \) so it must be projective. This is not true since \( Z = L \setminus \{x\} \simeq \mathbb{A}^1 \). \( Z \) admits nonconstant regular functions, so it cannot be projective. This contradiction shows that \( Y \) is not projective.

**Problem 4.** (Joins.) Let \( G(1,n) \) be the Grassmannian of lines in \( \mathbb{P}^n \) as in the previous homework. Show that:

(i) The set \( \{(L,P) : P \in L \} \subset G(1,n) \times \mathbb{P}^n \) is closed.

(ii) If \( Z \subset G(1,n) \) is any closed subset then the union of all lines \( L \subset \mathbb{P}^n \) such that \( L \in Z \) is closed in \( \mathbb{P}^n \).

(iii) Let \( X,Y \subset \mathbb{P}^n \) be disjoint projective varieties. Then the union of all lines in \( \mathbb{P}^n \) intersecting \( X \) and \( Y \) is a closed subset of \( \mathbb{P}^n \). It is called the join \( J(X,Y) \) of \( X \) and \( Y \).

**Answer:**

(i) We let

\[
J = \{(P,L) : P \in L \} \subset \mathbb{P}^n \times G(1,n).
\]

We will think of lines \( L \) in terms of their Plucker coordinates

\[
z_{ij} = a_i b_j - a_j b_i
\]

where \( a, b \) are two points on \( L \) with

\[
a = [a_0 : \ldots : a_n], \quad b = [b_0 : \ldots : b_n].
\]

In fact, it will be useful to form the vectors

\[
a = \sum a_i e_i, \quad b = \sum b_i e_i.
\]

Similarly, a point \( P \in \mathbb{P}^n \) will have an associated vector

\[
p = \sum p_i e_i.
\]

Now, if \( P \in L \), then \( p = sa + tb \) hence

\[
p \wedge a \wedge b = 0.
\]

Then

\[
\left( \sum p_i e_i \right) \wedge \left( \sum a_i e_i \right) \wedge \left( \sum b_i e_i \right) = \left( \sum p_i e_i \right) \wedge \left( \sum_{j<k} z_{jk} e_j \wedge e_k \right)
\]

\[
= \sum_{i<j<k} (p_i z_{jk} - p_j z_{ik} + p_k z_{ij}) e_i \wedge e_j \wedge e_k.
\]
The conclusion is that $J$ is defined by the equations

$$p_i z_{jk} - p_j z_{ik} + p_k z_{ij} = 0$$

which are bihomogeneous in the variables. Thus, $J$ is projective.

(ii) Let

$$p : J \rightarrow \mathbb{G}(1, n), q : J \rightarrow \mathbb{P}^n$$

be the natural projections. Then, for any $Z$ closed in $\mathbb{G}(1, n)$, the preimage $p^{-1}(Z)$ is also closed. Thus $q(p^{-1}(Z))$ is closed by the main theorem of projective varieties. This set consists in points $P$ lying on lines $L$ such that $L \in Z$, hence it can be identified with the union of all lines in $Z$.

(iii) We let $A$ be the set of lines intersecting $X$ and $B$ be the set of lines intersecting $Y$. We show $A$ and $B$ are closed in $\mathbb{G}(1, n)$, hence so is $Z = A \cap B$. The join $J(X, Y)$ is simply the union of lines contained in $Z$ hence it must be closed in $\mathbb{P}^n$ by item (ii).

It suffices to prove $A$ is closed in $\mathbb{G}(1, n)$. Indeed, we can think of $A$ as the projection under $p$ of the set

$$\{(P, L) : P \in L \cap X \times \mathbb{G}(1, n)) = J \cap q^{-1}(X).$$

Hence

$$A = p(J \cap q^{-1}(X))$$

which is closed because $p$ is closed and $q$ is continuous.

\[\square\]

**Problem 5.** (Rational varieties.) The definition of birational isomorphisms given in class extends to the projective category. Two projective varieties $X$ and $Y$ are birational if there are rational maps

$$f : X \dashrightarrow Y, \ g : Y \dashrightarrow X.$$

which are rational inverses to each other. Just as in the affine case, a birational isomorphism $f : X \dashrightarrow Y$ induces an isomorphism of the fields of rational functions $f^* : K(Y) \rightarrow K(X)$.

(i) Explain that if $X$ is rational, then

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X$ and $Y$ are rational, then $X \times Y$ is rational.

(iii) Show that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. 

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(iv) The group of automorphisms of the field of fractions in two variables $k(x, y)$ is called the Cremona group. Explain that the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$. Explain that the Cremona involution

$$(x, y) \mapsto (x^{-1}, y^{-1})$$
extends to an automorphism of $k(x, y)$. What is the corresponding birational involution of $\mathbb{P}^2$? Where is this birational automorphism regular?

(v) More generally, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group.

**Answer:** (i) Clearly, $\mathbb{A}^n$ is birational to $\mathbb{P}^n$, hence

$$K(\mathbb{P}^n) \cong K(\mathbb{A}^n) \cong k(t_1, \ldots, t_n).$$

Thus $X$ is rational iff

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Let $U \subset \mathbb{P}^n$ be the open set where the coordinate $x_0 \neq 0$. Similarly, let $V \subset \mathbb{P}^m$ be the open set where the coordinate $y_0 \neq 0$, and let $W$ be the open set in $\mathbb{P}^{n+m}$ where the first coordinate is non-zero. Define $\phi : U \times V \to \mathbb{P}^{n+m}$ as

$$[x_0 : x_1 : \ldots : x_n] \times [y_0 : y_1 : \ldots : y_m] \mapsto \left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \ldots : \frac{x_n}{x_0} : \frac{y_1}{y_0} : \frac{y_2}{y_0} : \ldots : \frac{y_m}{y_0}\right].$$

It is easy to check that $\phi$ establishes an isomorphism between $U \times V$ and $W$, with inverse

$$\psi : W \to U \times V, [1 : z_1 : \ldots : z_{n+m}] \to [1 : z_1 : \ldots : z_n] \times [1 : z_{n+1} : \ldots : z_{n+m}].$$

Therefore $\phi$ and $\psi$ define birational isomorphisms between $\mathbb{P}^n \times \mathbb{P}^m$ and $\mathbb{P}^{n+m}$. Finally, if $X$ and $Y$ are birational to $\mathbb{P}^n$ and $\mathbb{P}^m$, then $X \times Y$ is birational to $\mathbb{P}^n \times \mathbb{P}^m$ which in turn is birational to $\mathbb{P}^{n+m}$. Therefore, $X \times Y$ is rational.

(iii) Two closed subsets $\{a\} \times \mathbb{P}^1$ and $\{b\} \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ have nonempty intersection. This is false in $\mathbb{P}^2$ by problem 1(iii). Hence $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be isomorphic.

(iv) We explained in (i) that

$$K(\mathbb{P}^2) = k(x, y)$$

hence any automorphism of $k(x, y)$ corresponds to an automorphism of $K(\mathbb{P}^2)$ which in turn gives a birational isomorphism of $\mathbb{P}^2$. The involution

$$(x, y) \mapsto (x^{-1}, y^{-1})$$
corresponds to the birational map

$$f[x : y : z] = [x^{-1} : y^{-1} : z^{-1}].$$

This map is regular on $\mathbb{P}^2 \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Indeed, to show that the map is regular at the points where $(x, y) \neq (0, 0)$, we rewrite it in the form

$$f[x : y : z] = \left[\frac{z}{x} : \frac{z}{y} : 1\right].$$
(v) The automorphism
\[(x, y) \rightarrow (x^a y^b, x^c y^d)\]
where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})
\]
belongs to the Cremona group. Its inverse is
\[(x, y) \rightarrow (x^{a'} y^{b'}, x^{c'} y^{d'})\]
where
\[
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}
\]
is the inverse of the matrix above.

\[\square\]

**Problem 6.** *(Quadrics are rational.)* Show that any non-degenerate irreducible quadric \(Q \subset \mathbb{P}^{n+1}\) is birational to \(\mathbb{P}^n\).

**Answer:** All nondegenerate quadrics are isomorphic as seen in class. We can assume that the quadric \(Q\) is defined by
\[x_0 x_1 - x_2^2 - x_3^2 - \ldots - x_n^2 = 0.\]

Pick the point \(p = [1 : 0 : \ldots : 0] \in Q\), and let \(H\) be the hyperplane \(X_0 = 0\).

The line passing through \(p = [1 : 0 : \ldots : 0]\) and \(q = [x_0 : \ldots : x_n]\) is
\[[r + sx_0 : sx_1 : \ldots : sx_n].\]

This line intersects the hyperplane \(H\) when \(r + sx_0 = 0\). Therefore the line intersects \(H\) at \([0 : sx_1 : \ldots : sx_n] = [0 : x_1 : \ldots : x_n].\) This may be undefined when \(x_1 = x_2 = \ldots = x_n = 0\), i.e. when \(q = p\). We obtain a morphism
\[f : Q \setminus \{p\} \rightarrow H, \ [x_0 : x_1 : \ldots : x_n] \rightarrow [0 : x_1 : \ldots : x_n].\]

The rational inverse of \(f\) is given by
\[g : H \rightarrow Q, \ [x_1 : x_2 : \ldots : x_n] = \left[\frac{x_2^2 + \cdots x_n^2}{x_1} : x_1 : x_2 : \ldots : x_n\right].\]

This may be undefined at the points where \(x_1 = 0\).

Since \(f\) has a rational inverse, \(Q\) is birational to \(H\).