Math 203, Problem Set 1. Due Monday October 6.

For this problem set, you may assume that the ground field is algebraically closed.

1. Show that the Zariski topology on $\mathbb{A}^2$ is not the product of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

2. A topological space X is said to be Noetherian if it satisfies the ascending chain condition on open sets, i.e. any ascending chain of open sets eventually stabilizes.

(i) Check that any subset $Y \subset X$ of a Noetherian space is also Noetherian in the subspace topology.
(ii) Check that a Noetherian space which is also Hausdorff must be a finite set of points.
(iii) Show that a Noetherian space is *quasi-compact* i.e., show that any open cover of a Noetherian space has a finite subcover.
(iv) Show that $\mathbb{A}^n$ is Noetherian in the Zariski topology. Conclude that any affine algebraic set is Noetherian.

3. Let $\mathbb{A}^3$ be the 3-dimensional affine space with coordinates x, y, z. Find the ideals of the following algebraic sets:

(i) The union of the (x, y)-plane with the z-axis.
(ii) The image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \to (t, t^2, t^3)$. This is called the twisted cubic curve.

4. Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map *i.e.* $f(p) = (f_1(p), \ldots, f_m(p))$ for $p \in \mathbb{A}^n$, where $f_1, \ldots, f_m$ are polynomials in n variables. Are the following true or false:

(i) The image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ is an affine algebraic set.
(ii) The inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.
(iii) If $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph $\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic set.

5. Let X be the union of the three coordinate axes in $\mathbb{A}^3$. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements.

Remark: Note that X has dimension 1 even though it is cut out by 3 equations. We say that X is not a complete intersection.

6. Let X_1, X_2 be affine algebraic sets in $\mathbb{A}^n$. Assuming the Nullstellensatz if necessary, show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, $i.e.$ find affine algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$.