

Math 203, Problem Set 1. Due Wednesday October 7.

For this problem set, you may assume that the ground field is algebraically closed.

1. Show that the Zariski topology on \mathbb{A}^2 is not the product of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

- 2 (i) Check that any subset $Y \subset X$ of a Noetherian space is also Noetherian in the subspace topology.
(ii) Show that a Noetherian space is *quasi-compact* i.e., show that any open cover of a Noetherian space has a finite subcover.
(iii) Check that a Noetherian space which is also Hausdorff must be a finite set of points.
(iv) Show that \mathbb{A}^n is Noetherian in the Zariski topology. Conclude that any affine algebraic set is Noetherian.

3. Let \mathbb{A}^3 be the 3-dimensional affine space with coordinates x, y, z . Find the ideals of the following algebraic sets:

- (i) The union of the three coordinate axes.
(ii) The union of the z -axis and the (x, y) -plane.

4. Let $f : \mathbb{A}^n \rightarrow \mathbb{A}^m$ be a polynomial map i.e. $f(p) = (f_1(p), \dots, f_m(p))$ for $p \in \mathbb{A}^n$, where f_1, \dots, f_m are polynomials in n variables. Are the following true or false:

- (i) The image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ is an affine algebraic set.
(ii) The inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.
(iii) If $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph $\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic set.

5. Let X_1, X_2 be affine algebraic sets in \mathbb{A}^n . Assuming the Nullstellensatz if necessary, show that

- (i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, i.e. find affine algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$.

6. Find the irreducible components of the affine algebraic set $x^2 - yz = xz - x = 0$ in \mathbb{A}^3 . What is the dimension of this affine algebraic set?

7. Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in \mathbb{A}^3 . What is the dimension of this affine algebraic set?

8. Let X be the image of the map $\mathbb{A}^1 \rightarrow \mathbb{A}^3$ given by $t \rightarrow (t^3, t^4, t^5)$. Show that $I(X)$ cannot be generated by fewer than 3 elements.

Remark: Note that X has dimension 1 even though it is cut out by 3 equations. We say that X is not a *complete intersection*.