Math 203, Problem Set 2. Due Monday October 13.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Find the irreducible components of the affine algebraic set $x^2 - yz = xz - x = 0$ in $\mathbb{A}^3$. What is the dimension of this affine algebraic set?

2. Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in $\mathbb{A}^3$. What is the dimension of this affine algebraic set?

3. Prove that the irreducible components of a Noetherian topological space are unique. That is, if

$$X = \bigcup X_i = \bigcup Y_i$$

such that all sets X_i and Y_i are irreducible, assumed to be irredundant (e.g. $X_i \not\subset X_j$ and similarly $Y_i \not\subset Y_j$ for $i \neq j$) then X_i's are a permutation of the Y_i's.

4. Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$ whose zero locus $\mathcal{Z}(f)$ in $\mathbb{A}^2$ is not irreducible.

5. Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

6. An algebraic set $\mathcal{Z} \subset \mathbb{A}^2$ defined by an irreducible polynomial f of degree 2 is called an irreducible conic. Show that any irreducible conic can be written in the form

$$Y - X^2 = 0 \text{ or } XY - 1 = 0$$

after an affine change of coordinates in $\mathbb{A}^2$.

Remark: An affine change of coordinates taking (x, y) into (X, Y) is a transformation of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b},$$

where A is a 2×2 invertible matrix and $\mathbf{b} \in \mathbb{A}^2$ is a vector.

7. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using the previous problem, show that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

8. (*Baby Bezout Theorem*) Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be distinct algebraic subsets of $\mathbb{A}^2$ given as

$$\mathcal{Z}_1 = \mathcal{Z}(f),\ \mathcal{Z}_2 = \mathcal{Z}(g)$$

where $f, g \in k[X, Y]$ are irreducible polynomials of degrees d and e. Show that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ intersect in de points, or equivalently that f and g have at most de common zeros.

(i) Prove that the intersection $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is finite.

Hint: If $(a, b) \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, then $f(X, b)$ and $g(X, b)$ have a common factor $X - a$. Conclude that b must be a root of $R_{f,g}$, when viewing f and g as polynomials in X with coefficients in $k[Y]$. Show that b can only take on only finitely many values.

(ii) Let
$$F(X,Y) = f(X, Y + \lambda X), G(X,Y) = g(X, Y + \lambda X),$$
for suitable scalar $\lambda \in k$. Show that F and G have degrees at most d and e. Show that F and G are distinct irreducible polynomials. Show that we may pick λ such that that $\mathcal{Z}(F) \cap \mathcal{Z}(G)$ consists in n points
$$\{(a_1, c_1), \ldots, (a_n, c_n)\}$$
where
$$c_i \neq c_j \text{ for all } i \neq j.$$

(iii) Writing
$$F(X,Y) = F_d(Y)X^d + \ldots + F_0(Y), \quad G(X,Y) = G_e(Y)X^e + \ldots + G_0(Y)$$
as polynomials in X with coefficients in $k[Y]$, show that the resultant $R_{F,G} \in k[Y]$ has degree at most de. Conclude that if $(a,c) \in \mathcal{Z}(F) \cap \mathcal{Z}(G)$, then c can take on at most de values. Show that this implies that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ has at most de intersection points.