Math 203, Problem Set 2. Due Wednesday October 21.

For this problem set, you may assume that the ground field is algebraically closed.

1. Which of the following algebraic sets are isomorphic:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2$.

2. (*Products.*) *Part I.* Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine algebraic sets.

(i) Show that $X \times Y$ is an affine algebraic set (in such a fashion that its Zariski topology is the subspace topology from $\mathbb{A}^{n+m}$).
(ii) Show that if X and Y are irreducible, then $X \times Y$ is also irreducible.

Hint: Let $X \times Y = Z_1 \cup Z_2$, with Z_1 and Z_2 closed. Show that $X_i = \{x \in X : x \times Y \subset Z_i\}$ are closed and cover X. Conclude that $X_1 = X$ or $X_2 = X$, hence $Z_1 = X \times Y$ or $Z_2 = X \times Y$.

(iii) Show that $X \times Y$ is *a product in the category of affine algebraic sets* e.g it satisfies the following universal proprety: there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for every affine algebraic set Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $h : Z \to X \times Y$ such that $h \circ \pi_X = f$ and $h \circ \pi_Y = g$.

Part II. Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively.

(i) Construct the product prevariety $X \times Y$ by glueing the affine varieties $U_i \times V_j$ together.
(ii) Show that the universal property for products stated in I.(ii) still holds. Conclude that the product $X \times Y$ is independent of choices.
(iii) Show that if X and Y are varieties then $X \times Y$ is also a variety.

3. (*Isomorphisms of the affine and projective line.)*

(i) Show that every isomorphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ is of the form $f(x) = ax + b$.
(ii) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$f(x) = \frac{ax + b}{cx + d}$$

for some $a, b, c, d \in k$, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.
(iii) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

4. (*Conics.*) An algebraic set $C \subset \mathbb{A}^2$ defined by an irreducible polynomial f of degree 2 is called an irreducible conic.

(i) Show that any irreducible conic is isomorphic to one of the following

$$Y - X^2 = 0 \text{ or } XY - 1 = 0.$$

You may want to start with an arbitrary conic and complete the square several times.

(ii) Let C_1 and C_2 be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that C_1 and C_2 intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

5. (*Weak Bezout Theorem*) Let X and Y be distinct algebraic subsets of $\mathbb{A}^2$ given as

$$X = \mathcal{Z}(f),\ Y = \mathcal{Z}(g)$$

where $f, g \in k[X, Y]$ are irreducible polynomials of degrees d and e. Show that $X \cap Y$ intersect in de points, or equivalently that f and g have at most de common zeros.

(i) Prove that the intersection $X \cap Y$ is finite.

Hint: If $(a, b) \in X \cap Y$, then $f(X, b)$ and $g(X, b)$ have a common factor $X - a$. Conclude that b must be a root of $R_{f,g}$, when viewing f and g as polynomials in X with coefficients in $k[Y]$. Show that b can only take on only finitely many values.

(ii) Let

$$F(X, Y) = f(X, Y + \lambda X), G(X, Y) = g(X, Y + \lambda X),$$

for suitable scalar $\lambda \in k$. Show that F and G have degrees at most d and e. Show that F and G are distinct irreducible polynomials. Show that we may pick λ such that that $\mathcal{Z}(F) \cap \mathcal{Z}(G)$ consists in n points

$$\{(a_1, c_1), \ldots, (a_n, c_n)\}$$

where

$$c_i \neq c_j \text{ for all } i \neq j.$$

(iii) Writing

$$F(X, Y) = F_d(Y)X^d + \ldots + F_0(Y), \quad G(X, Y) = G_e(Y)X^e + \ldots + G_0(Y)$$

as polynomials in X with coefficients in $k[Y]$, show that the resultant $R_{F,G} \in k[Y]$ has degree at most de. Conclude that if $(a, c) \in \mathcal{Z}(F) \cap \mathcal{Z}(G)$, then c can take on at most de values. Show that this implies that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ has at most de intersection points.

(iv) Show that if $X, Y \subset \mathbb{P}^2$ are distinct projective algebraic sets defined by the vanishing of two homogeneous irreducible polynomials of degrees d and e, then $X \cap Y$ intersect in at most de points. You may want to reduce to the case you already proved after changing coordinates.