Math 203, Problem Set 3. Due Monday October 20.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Which of the following algebraic sets are isomorphic:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2$.

2. (*Hartogs theorem and quasi-affine algebraic sets.*) Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not isomorphic to an affine algebraic set:

(i) Prove first the following algebro-geometric analogue of Hartog's theorem in complex analysis: if $f : X \to k$ is a regular function on X, then f must be a polynomial in $k[x, y]$.

Hint: Write $f = g/h$ for polynomials g, h without common factors. Observe that h cannot vanish on $\mathbb{A}^2 \setminus \{(0,0)\}$, and conclude that h must be constant.

(ii) If $\Phi : Y \to X$ is an isomorphism between an affine algebraic set Y and X, consider the composition $\Psi = \iota \circ \Phi : Y \to \mathbb{A}^2$, where $\iota : X \to \mathbb{A}^2$ is the inclusion. Show that the morphism Ψ induces an isomorphism on coordinate rings. Conclude that Ψ must be an isomorphism, hence ι must be an isomorphism, which must be a contradiction.

3. (*Products.*) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine algebraic sets.

(i) Show that $X \times Y$ is an affine algebraic set (in such a fashion that its Zariski topology is the subspace topology from $\mathbb{A}^{n+m}$).
(ii) Show that if X and Y are irreducible, then $X \times Y$ is also irreducible.

Hint: Let $X \times Y = Z_1 \cup Z_2$, with Z_1 and Z_2 closed. Show that $X_i = \{x \in X : x \times Y \subset Z_i\}$ are closed and cover X. Conclude that $X_1 = X$ or $X_2 = X$, hence $Z_1 = X \times Y$ or $Z_2 = X \times Y$.

(iii) Show that $X \times Y$ is *a product in the category of affine algebraic sets* e.g it satisfies the following universal proprety: for every affine algebraic set Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $h : Z \to X \times Y$ such that $h \circ \pi_X = f$ and $h \circ \pi_Y = g$.
(iv) Conclude that

$$A(X \times Y) = A(X) \otimes_k A(Y).$$

4. Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with n elements in $\mathbb{A}^1$.

(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \to \mathbb{A}^1 \setminus S,\ (X_0, \dots, X_n) \mapsto X_0$$

is an isomorphism.

(ii) Show that the ring of regular functions on $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to $k\left[t, \frac{1}{t}\right]$, the ring of polynomials in t and $\frac{1}{t}$.

(iii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

Hint: Assume that

$$\Phi : A(X) \to k\left[t, t^{-1}\right]$$

is an isomorphism. Observe that X_i are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers m_i. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of i.

5. Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \to Z_n,\ f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n,\ \ g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and W_n where g becomes an isomorphism.

(ii) Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism between $Z_n \to W_n$.

(iii) Assume that there exists an isomorphism

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) Repeat the argument above without the assumption that h sends the origin to itself. You may need to prove a stronger version of Problem 4.

(v) Show that Z_1 and W_1 are isomorphic. Write down an isomorphism between them.

6. (*Cubic curves are not rational.*) Let $\lambda \in k \setminus \{0,1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)}\right)$$

where the pairs of polynomials (f,g) and (p,q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. Conclude that $f, g, f-g, g-\lambda g$ must be perfect squares.

(ii) Prove the following:

Lemma: If f, g are polynomials in $k[t]$ without common factors and such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f-g, f-\lambda g$ are perfect squares, then f and g must be constant.

Hint: Pick f and g such that $\max(\deg f, \deg g)$ is minimal among all pairs (f,g) which satisfy the requirement that $f, g, f-g, f-\lambda g$ are squares for some $\lambda \neq 0, 1$. Write $f = u^2, g = v^2$. Considering the factorization of $f-g$ and $f-\lambda g$, prove that $u, v, u-\mu v, u+\mu v$ are also squares for some constant $\mu \neq 0, 1$. Conclude that $\max(\deg u, \deg v) < \max(\deg f, \deg g)$ unless f, g are constant. Why is this a contradiction?

Remark: We will see later that any cubic curve can be written in the form

$$y^2 - x(x-1)(x-\lambda) = 0, \text{ or } y^2 - x^3 = 0 \text{ or } y^2 - x^2(x-1) = 0, .$$

The latter curves are Z_2 and W_2 in Problem 4, so they are birational to $\mathbb{A}^1$.