Math 203, Problem Set 2. Due Monday, November 9.

For this problem set, you may assume that the ground field is algebraically closed.

1. (*Conics through* 5 *points*.)

(i) Four points in $\mathbb{P}^2$ are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if $p_1, \ldots, p_4$ are points in general position, there exists a linear change of coordinates

$$T : \mathbb{P}^2 \to \mathbb{P}^2$$

with

$$T([1:0:0]) = p_1,\ T([0:1:0]) = p_2,\ T([0:0:1]) = p_3,\ T([1:1:1]) = p_4.$$

(ii) Given five distinct points in $\mathbb{P}^2$, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part (i) to assume that four of the points are $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$.

2. *(Twisted curves and complete intersections.)*

A variety Y of dimension r in $\mathbb{P}^n$ is a *strict complete intersection* if the ideal $I(Y)$ can be generated by $n-r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n-r$ hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.

(ii) Show that the twisted cubic T in $\mathbb{P}^3$ can be written as the set-theoretic intersection of the quadric and the cubic

$$Q = \mathcal{Z}(y^2 - xz), C = \mathcal{Z}(z^3 + xw^2 - 2yzw).$$

In particular, T is a set theoretic complete intersection.

(iii) Show that T is the intersections of the three quadrics

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xw - yz), Q_3 = \mathcal{Z}(yw - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic. In particular $I(T)$ contains three elements of degree 2.

(iv) Possibly using (iii), explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence T is not a strict complete intersection.

Remark: It is an unsolved problem to show that every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic complete intersection. This is called the *Hartshorne conjecture*.

3. (*Introduction to moduli theory.*) Show that for any 3 lines L_1, L_2, L_3 in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric Q and not on the polynomial defining it. Let us denote this point by p_Q. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

Remark: The projective space $\mathbb{P}^9$ is said to be the moduli space of quadrics in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subpace

$$H_L \subset \mathbb{P}^9$$

such that

$$L \subset Q \text{ iff and only if } p_Q \in H_L.$$

Hint: You may want to change coordinates to assume that the line L is cut out by the equations $X_0 = X_1 = 0$. This will force 3 of the coefficients of Q to be zero. Which ones? What is the subspace H_L in this case?

(iii) Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that L_1, L_2, L_3 are contained in a quadric Q.

4. (*Grassmannians.*) We will make the space $G(1,n)$ of all lines in $\mathbb{P}^n$ into a projective variety. We define a set-theoretic map

$$\phi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with

$$N = \binom{n+1}{2} - 1$$

as follows. For every line $L \subset \mathbb{P}^n$, choose two distinct points

$$P = (a_0 \dots a_n) \text{ and } Q = (b_0 \dots b_n)$$

on L and define $\phi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the maximal minors of the matrix

$$\begin{pmatrix} a_0 : \dots : a_n \\ b_0 : \dots : b_n \end{pmatrix}$$

in any fixed order. These are called the Plucker coordinates of L. The map ϕ is called the Plucker embedding.

(i) Prove that the map ϕ is well-defined and injective.

(ii) The image of ϕ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. You may want to recall the Gaussian algorithm which brings *almost* any matrix as above into the form

$$\begin{pmatrix} 1 \ 0 \ a'_2 \dots a'_n \\ 0 \ 1 \ b'_2 \dots b'_n \end{pmatrix}.$$

(iii) Show that $G(1,1)$ is a point, $G(1,2) = \mathbb{P}^2$, and $G(1,3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.