

Math 203, Problem Set 4. Due Monday October 27.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. (Products of prevarieties.) Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively.

- (i) Construct the product prevariety $X \times Y$ by glueing the affine varieties $U_i \times V_j$ together.
- (ii) Show that there are projection morphisms $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ satisfying the usual universal property for products: given morphisms $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ from any prevariety Z , there is a unique morphism $h : Z \rightarrow X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$. Deduce that the product constructed in (i) is independent of choices.
- (iii) Show that if X and Y are varieties then $X \times Y$ is also a variety.

2. (Rational functions on prevarieties.) Let X be a prevariety. Consider pairs (U, f) where U is an open subset of X and $f \in \mathcal{O}_X(U)$ a regular function on U . We call two such pairs (U, f) and (U', f') equivalent if there is a nonempty open subset $V \subset U \cap U'$ such that

$$f|_V = f'|_V.$$

- (i) Show that this defines an equivalence relation.
- (ii) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the field of rational functions on X and denoted $K(X)$.
- (iii) If X is an affine variety, show that $K(X)$ is just the field of rational functions as defined in class.

3. (Isomorphisms of the affine and projective line.)

- (i) Show that every isomorphism $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is of the form $f(x) = ax + b$.
- (ii) Show that every isomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in k$, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.
- (iii) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

4. (Rational curves.) Let $X \subset \mathbb{P}^2$ be a projective variety. A morphism $f : \mathbb{P}^1 \rightarrow X$ is a polynomial map

$$f([x : y]) = (f_0([x : y]), f_1([x : y]), f_2([x : y])),$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree, such that $f(\mathbb{P}^1) \subset X$.

Prove the following facts about lines and conics in projective plane:

- (i) For any line $L \subset \mathbb{P}^2$, there is a bijective morphism

$$f : \mathbb{P}^1 \rightarrow L.$$

- (ii) For any irreducible conic $C \subset \mathbb{P}^2$, there is a bijective morphism

$$f : \mathbb{P}^1 \rightarrow C.$$

You may wish to change coordinates so that your conic has a convenient expression.

Remark: A curve X which is the image of a morphism $f : \mathbb{P}^1 \rightarrow X \subset \mathbb{P}^2$ is called a rational curve in X . This is the same thing as saying that X admits a polynomial parametrization. In particular, we showed that lines and conics in \mathbb{P}^2 are rational curves in \mathbb{P}^2 .

(iii) Consider the elliptic curve $\overline{E}_\lambda \subset \mathbb{P}^2$:

$$y^2z = x(x-z)(x-\lambda z).$$

Show that there are no nonconstant morphisms

$$\mathbb{P}^1 \rightarrow \overline{E}_\lambda \subset \mathbb{P}^2.$$

Therefore, elliptic curves are not rational curves.

5. (*Conics through 5 points.*) *Part A.* Show that a line and an irreducible conic in \mathbb{P}^2 cannot intersect in 3 points.

Part B.

(i) Four points in \mathbb{P}^2 are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if p_1, \dots, p_4 are points in general position, there exists a linear change of coordinates

$$T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$$

with

$$T([1 : 0 : 0]) = p_1, \quad T([0 : 1 : 0]) = p_2, \quad T([0 : 0 : 1]) = p_3, \quad T([1 : 1 : 1]) = p_4.$$

(ii) Given five distinct points in \mathbb{P}^2 , no three of which are collinear, show that there is a unique irreducible projective conic passing through all five points. You may want to use part (i) to assume that four of the points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$.

(iii) Deduce that two distinct irreducible conics in \mathbb{P}^2 cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

Remark: For any degree d , fix $3d - 1$ points in \mathbb{P}^2 in “general position”. You may ask how many rational curves of degree d in \mathbb{P}^2 pass through these $3d - 1$ points. Clearly, there is $N_1 = 1$ line through 2 points, and we have shown that $N_2 = 1$ conic through 5 points. The next few numbers are

$$N_3 = 12, N_4 = 620, N_5 = 87,304, N_6 = 26,312,976, N_7 = 14,616,808,192.$$

Thus, there are 12 rational cubics through 8 points, 620 rational quartics through 11 points and so on. A general answer for arbitrary d was found in 1994 using ideas from physics/string theory. The area of algebraic geometry that computes these numbers is called enumerative geometry/Gromov-Witten theory.

6. (*Grassmannians.*) We will make the space of all lines in \mathbb{P}^n into a projective variety. We define a set-theoretic map

$$\phi : \{\text{lines in } \mathbb{P}^n\} \rightarrow \mathbb{P}^N$$

with

$$N = \binom{n+1}{2} - 1$$

as follows. For every line $L \subset \mathbb{P}^n$, choose two distinct points

$$P = (a_0 \dots a_n) \text{ and } Q = (b_0 \dots b_n)$$

on L and define $\phi(L)$ to be the point in \mathbb{P}^N whose homogeneous coordinates are the maximal minors of the matrix

$$\begin{pmatrix} a_0 & \dots & a_n \\ b_0 & \dots & b_n \end{pmatrix}$$

in any fixed order. Show that:

- (i) The map ϕ is well-defined and injective. The map ϕ is called the Plucker embedding.
- (ii) The image of ϕ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. You may want to recall the Gaussian algorithm which brings *almost* any matrix as above into the form

$$\begin{pmatrix} 1 & 0 & a'_2 & \dots & a'_n \\ 0 & 1 & b'_2 & \dots & b'_n \end{pmatrix}.$$

- (iii) Show that $G(1, 1)$ is a point, $G(1, 2) = \mathbb{P}^2$, and $G(1, 3)$ is the zero locus of a quadratic equation in \mathbb{P}^5 .