Math 203, Problem Set 1. Due Wednesday, April 18.

1. (*Examples of flat morphisms.*)

(i) Give an example of a flat morphism $f : X \to T$ such that $f^{-1}(t)$ is irreducible for all $t \neq t_0$, while $f^{-1}(t_0)$ is not irreducible.

(ii) Give an example of a flat morphism $f : X \to T$ such that $f^{-1}(t)$ is reduced for all $t \neq t_0$, while $f^{-1}(t_0)$ is not reduced.

2. (*Flatness and pullbacks.*) Assume $f : X \to Y$ is a flat morphism. Show that exact sequences of coherent sheaves on Y are still exact on X after pull back.

3. (*Base-change and flatness.*)

(i) Show that if M is an A-flat module, and B is A-algebra, then $M \otimes_A B$ is B-flat.

(ii) Show that flatness $f : X \to Y$ is preserved under base-change by a morphism $g : Y' \to Y$.

4. (*Flatness and sections.*)

(i) Show that if M is B-flat and B is a flat A-algebra, then M is A-flat.

(ii) Let A be a B-algebra, and let M be A-module. Show that M is B-flat iff $M_{\mathfrak{p}}$ is B-flat for all primes $\mathfrak{p}$ in A.

(iii) Let $f : X \to Y$ be flat and $U \subset X$ and $V \subset Y$ are affine open sets with $U \subset f^{-1}(V)$. Show the flatness of the homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U).$$

5. (*Flatness over PID, DVR, dual numbers.*)

(i) Show that a module M is A-flat iff $\operatorname{Tor}_1^A(M, A/\mathfrak{i}) = 0$ for all ideals $\mathfrak{i}$ in A.

(ii) Show that if A is a PID, then M if A-flat iff M is torsion free.

(iii) Show that if A is a DVR with local parameter t, then M is A-flat iff t is not a zero divisor on M.

(ii) Let $D = k[\epsilon]/(\epsilon^2)$. Show that M is D-flat iff the multiplication by $\epsilon : M/\epsilon M \to M$ is injective. This is important in deformation theory.

6. (*Example of a non-flat morphism.*) Let

$$X = \text{Spec } k[x, y, z, w]/(xz, yz, xw, yw)$$

be the union of the (xy)-plane and (zw)-plane in $\mathbb{A}^4$. Let

$$\pi : X \to \mathbb{A}^2$$

be given by

$$k[s, t] \to k[x, y, z, w]/(xz, yz, xw, yw), \quad s \to x + z, t \mapsto y + w.$$

Show that π is not flat. You may wish to check the definition directly.

Geometrically, X is a union of two planes intersecting at a point, and π maps these planes isomorphically onto $\mathbb{A}^2$. This example should be contrasted with the criterion for flatness discussed in class when the base is a smooth curve.

The final question is *optional*.

7. (*Some deformation theory. The Hilbert "scheme".*)
Let

$$\mathcal{F} : (Sch) \to (Set)$$

be a contravariant functor between the category of schemes and the category of sets.

Let X be an element of the set $\mathcal{F}(\text{Spec } k)$. The tangent space of $\mathcal{F}$ at X is by definition the fiber over X of the natural map

$$\mathcal{F}(\mathbb{D}) \to \mathcal{F}(\text{Spec } k).$$

Here $\mathbb{D} = \text{Spec } k[\epsilon]/(\epsilon^2)$.

(i) Let X be a variety over k. Recall that in the first few lectures of Math 203b we have defined the functor of points

$$h_X : (Sch) \to (Set), \quad S \mapsto \text{Mor}(S, X).$$

Note that $h_X(\text{Spec } k) = X(k)$. Show that the tangent space of the functor h_X at the point $p \in X$ coincides with the Zariski tangent space $T_{X,p}$. This should be immediate provided you solved an older homework in Math 203b.

(ii) The Hilbert functor

$$\mathcal{H} : (Sch) \to (Sets)$$

is defined by assigning to each scheme S, flat families

$$\pi : \mathcal{X} \to S, \quad \mathcal{X} \subset S \times \mathbb{P}^r$$

up to isomorphism of S-schemes. Using problem 3(ii), convince yourself that $\mathcal{H}$ is a contravariant functor.

(iii) If $X \subset \mathbb{P}^r$, then the tangent space of the Hilbert functor at X is $\text{Hom}_{\mathbb{P}^r}(\mathcal{I}_X, \mathcal{O}_X)$. This tangent space parametrizes infinistesimal deformations of X inside $\mathbb{P}^r$ i.e. $\mathbb{D}$-flat families

$$\mathcal{X} \to \mathbb{D}, \quad \mathcal{X} \subset \mathbb{P}^r \times \mathbb{D},$$

which restrict to X.

Hint: You should first treat the affine case, then glue. For the affine case, the problem can be rephrased as follows: given an ideal

$$\mathfrak{i} \subset A$$

find an ideal

$$\mathfrak{i}' \subset A' := A[\epsilon]/(\epsilon^2)$$

with $A'/\mathfrak{i}'$ flat over D, such that

$$A'/\mathfrak{i}' \otimes_D k = A/\mathfrak{i}.$$

Show these ideals are classified by $Hom_A(\mathfrak{i}, A/\mathfrak{i})$. To this end, the criterion of flatness in Problem 5 gives the exactness of

$$0 \to A/\mathfrak{i} \to^{\times\epsilon} A'/\mathfrak{i}' \to A/\mathfrak{i} \to 0.$$

Use this as a first row of a natural square diagram and diagram chase.

If you get suck, you can look this up in Hartshorne's deformation theory, but do write up your own solution.

(iii) In particular, if X is smooth, the tangent space can be expressed as

$$\mathrm{Hom}_{\mathbb{P}^r}(\mathcal{I}_X, \mathcal{O}_X) = \mathrm{Hom}_X(\mathcal{I}_X/I_X^2, \mathcal{O}_X) = \mathrm{Hom}_X(N_{X/\mathbb{P}^r}^\vee, \mathcal{O}_X) = H^0(X, N_{X/\mathbb{P}^r}).$$

This is of course quite suggestive if you draw a picture.

(iv) It is a difficult result that the Hilbert functor is the functor of points of a certain scheme, called the Hilbert scheme. The above calculation therefore gives the tangent space to the Hilbert scheme, and it is useful to get an upper bound on the dimension.