Math 203, Problem Set 2. Due Wednesday, May 2.

1. (*Mayer-Vietoris.*) Show that if Y_1, Y_2 are two closed subsets of X, there is an exact sequence

$$A_k(Y_1 \cap Y_2) \to A_k(Y_1) \oplus A_k(Y_2) \to A_k(Y_1 \cup Y_2) \to 0.$$

2. (*Blowups.*) Compute the Chow groups of the blowup of $\mathbb{P}^2$ at n points.

3. (*Kunneth decomposition.*)

(i) Show that there are well-defined exterior products

$$A^k(X) \otimes A^\ell(Y) \to A^{k+\ell}(X \times Y)$$

which send

$$[Z] \otimes [W] \mapsto [Z \times W].$$

(ii) Show that if X admits an affine decomposition, then

$$\bigoplus_{k+\ell=m} A^k(X) \otimes A^\ell(Y) \to A^m(X \times Y)$$

is surjective. By contrast with algebraic topology, in general, this map is neither injective nor surjective.

4. (*Product of projective spaces.*)

(i) Compute the Chow groups of $X = \mathbb{P}^n \times \mathbb{P}^m$. By Problem 3, you should find generators given by products of linear spaces. To show that there are no relations you may wish to use projections via proper morphisms.

(ii) If Y is a hypersurface of bidegrees (d, e) in X, show that $[Y] = dh_1 + eh_2$ in $A^1(X)$, where h_1 and h_2 are the two hyperplane classes on X.

5. (*Counting fixed points.*) In this problem, we need to assume that there is an intersection product for $X = \mathbb{P}^n \times \mathbb{P}^m$. That is, the assignment

$$[V] \cdot [W] = [V \cap W]$$

if V, W are subvarieties intersecting transversally, makes

$$A^\star(X) = \oplus_k A^k(X)$$

into a ring. This fact holds true for any smooth X.

(i) Formulate a version of Bezout's theorem for $\mathbb{P}^n \times \mathbb{P}^m$ giving the number of intersection points of $(n+m)$ hypersurfaces of bidegrees (d_i, e_i) for $1 \leq i \leq n+m$.

(ii) Show that if $f : \mathbb{P}^n \to \mathbb{P}^n$ is a degree d morphism, then the class of the graph of f in $A^\star(\mathbb{P}^n \times \mathbb{P}^n)$ equals

$$[\Gamma_f] = \sum d^k \cdot h_1^k \cdot h_2^{n-k}.$$

Derive the class of the diagonal Δ in $A^\star(X)$ in terms of the generators h_1, h_2.

Hint: By Problem 4 you should be able to write

$$[\Gamma_f] = \sum_{i+j=n} c_i h_1^i h_2^j.$$

To find the coefficients c_{ij} you may wish to intersect with complementary classes.

(iii) Find the number of fixed points of a morphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree d.

The last problem is *optional*. It describes Hironaka's example of a proper non-projective variety.

Remark: This a source of further counterexamples. With suitable modifications, Hironaka's construction yields examples of schemes whose quotients by free actions of finite groups are not schemes. (This is reconciled in the land of *algebraic spaces*, which we will not cover in this course).

6. (*Example of a proper non-projective variety.*) Let $X = \mathbb{P}^3$ and consider C_1, C_2 two smooth *rational* curves intersecting in 2 points P_1 and P_2. If you want a concrete example, you can take

$$C_1 = \{x_3 = x_2 - x1 = 0\},\ \ C_2 = \{x_3 = x_0x_2 - x_1^2 = 0\}.$$

Let $\tilde{X}_1' \to X$ be the blowup of X at C_1, and let $\tilde{X}_1 \to \tilde{X}_1'$ denote the blow-up at the strict transform of C_2. Denote by

$$\pi_1 : \tilde{X}_1 \to X$$

the projection map. Similarly, let $\tilde{X}_2 \to X$ be the composition of the two blow-ups in the opposite order; first blow up C_2 and then the strict transform of C_1. Obviously, $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic away from the inverse image of $\{P1, P2\}$ so we can glue $\pi_1^{-1}(X \setminus \{P_1\})$ and $\pi_2^{-1}(X \setminus \{P_2\})$ along the isomorphism

$$\pi_1^{-1}(X \setminus \{P_1, P_2\}) = \pi_2^{-1}(X \setminus \{P1, P2\})$$

to get our example Y. From the construction there is an obvious projection map $\pi : Y \to X$.

(i) You can start by drawing a picture to see what's going on.
(ii) Show that Y is proper. To this end, prove first that properness is affine local on the base. Then you would reduce the problem to a statement about blowups which should be fairly clear.
(iii) Understand that the preimage $\pi^{-1}(P_1)$ consists of two curves L_1' and L_1'' arising from the two blowups. Similarly, $\pi^{-1}(P_2)$ consists in two curves L_2', L_2''.

(iv) Let L_1, L_2 be the preimages under π of two points Q_1 and Q_2 over C_1, C_2 respectively which are not the intersection points. Convince yourselves that the following are rationally equivalent

$$L_1 \equiv L_1' + L_1'', \;\; L_2 \equiv L_2' + L_2''$$
$$L_1'' \equiv L_2, \quad L_2'' \equiv L_1.$$

Conclude that you have found two effective curves D_1, D_2 such that

$$[D_1] + [D_2] = 0$$

in $A_1(Y)$. If Y is assumed projective, intersect with an ample class to get a contradiction.