Math 203, Problem Set 3. Due Friday, May 11.

1. (*Intersection product on surfaces.*)

Let C, D are two irreducible curves on a smooth surface X intersecting transversally. Show that

$$\deg(C \cdot D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D)).$$

Remark: In fact, the assumption of transversal intersection is not necessary. You can read the argument in Hartshorne Chapter 5, Proof of Theorem 1.1 to see how to reduce to this case.

Hint: Use

$$0 \to \mathcal{O}_X(-C-D) \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_{C \cap D} \to 0.$$

2. (*Divisor calculations on surfaces.*)

(i) Let C be a smooth curve of genus g. Show that if Δ is the diagonal in $C \times C$ then $\Delta^2 = 2 - 2g$.

Hint: You can either use the genus formula for curves on surfaces or alternatively, a normal bundle computation.

(ii) Show that if X is a degree d surface in $\mathbb{P}^3$, compute the genus of a smooth hyperplane section $C = X \cap H$ of X.

Remark: In particular, this should offer a different method of solving Problem 3(iii), Problem Set 5 of Math 203B.

(iii) If X is a degree d smooth surface in $\mathbb{P}^3$ show that $K_X^2 = d(d-4)^2$.

3. (*Flat pullbacks.*)

If $f : X \to Y$ is flat and $E \to X$ is a vector bundle, then starting from the definitions show that for all $\alpha \in A_\star(Y)$ we have

$$s_i(f^\star E) \cdot f^\star \alpha = f^\star(s_i(E) \cdot \alpha).$$

4. (*Self study: Whitney formula.*)

From Gathmann, read Proposition 10.3.1 and write your own proof of this result. The statement reads that if $0 \to E' \to E \to E'' \to 0$ then

$$c(E) = c(E')c(E'').$$

Please justify each step in your argument.

5. (*Hirzebruch surfaces.*)

Consider the bundle $E_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ over the projective line $X = \mathbb{P}^1$. The Hirzebruch surface is defined as

$$\mathbb{F}_n = \mathbb{P}(E_n).$$

It comes equipped with a projection morphism

$$p : \mathbb{F}_n \to X.$$

Clearly, p is a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$.

(i) Show that all fibers $p^{-1}(x)$ for $x \in X$ are rationally equivalent 1-cycles on $\mathbb{F}_n$. Write D for the corresponding Chow class in $A_1(\mathbb{F}_n)$.

(ii) Let

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$$

be the natural inclusion of the first factor. Projectivizing, show that this determines a morphism

$$s : X \to \mathbb{P}(E_n) = \mathbb{F}_n$$

such that

$$p \circ s = \mathbf{1}, \;\; s^{\star}\mathcal{O}_{\mathbb{F}_n}(-1) = \mathcal{O}_{\mathbb{P}^1},$$

where as before $O_{\mathbb{F}_n}(-1)$ denotes the tautological bundle over $\mathbb{F}_n = \mathbb{P}(E_n)$. Write C for the image of s, which is a curve on $\mathbb{F}_n$.

(iii) Possibly using a cellular decomposition, show that

$$A_0(\mathbb{F}_n) = \mathbb{Z}, \;\; A_1(\mathbb{F}_n) = \mathbb{Z}[C] \oplus \mathbb{Z}[D], \;\; A_2(\mathbb{F}_n) = \mathbb{Z}.$$

(iv) Find all intersection products C^2, D^2, and $C \cdot D$. Formulate a Bezout theorem for the surface $\mathbb{F}_n$.

Hint: One way to find C^2 is via a normal bundle computation. Begin with the defining exact sequence

$$0 \to T_C \to T_{\mathbb{F}_n}|_C \to N_{C/\mathbb{F}_n} \to 0.$$

To find $T_{\mathbb{F}_n}$, or rather the relative tangent bundle $T_{\mathbb{F}_n/X}$, recall the Euler sequence from Math 203b, PSet 5.

(v) Write C' for the curve obtained by the construction in (ii) by using the second factor inclusion

$$0 \to \mathcal{O}_{\mathbb{P}^1}(n) \to E_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n).$$

What is C'^2? Express C' in terms of C and D.