Math 203, Problem Set 5. Due Wednesday, May 31.

1. (*HRR for surfaces.*) Let X be a smooth projective surface, and let $L \to X$ be a line bundle. HRR states

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{L \cdot (L - K_X)}{2}.$$

In this problem, we verify HRR directly.

(i) Assume $L = \mathcal{O}_X(C)$ where C is a *smooth* curve on X. Use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0$$

to show that

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \chi(\mathcal{O}_X(C)|_C).$$

Use Riemann-Roch for the curve C and the line bundle $\mathcal{O}_X(C)|_C$ to show that

$$\chi(\mathcal{O}_X(C)|_C) = 1 - g_C + C^2.$$

Use the genus formula to find g_C in terms of C and K_X. Substitute and confirm HRR in this case.

(ii) Carry out the same argument for $L = \mathcal{O}_X(C - D)$, when C, D are *smooth* curves on X.

It can be shown (Hartshorne, Ch V, Section 1) that any line bundle $L \to X$ can be written in this form, so this establishes HRR for all line bundles over surfaces.

Hint: Use the following exact sequences

$$0 \to \mathcal{O}_X(C - D) \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_D \to 0$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0.$$

2. (*HRR for threefolds.*) Let X be a smooth projective 3-fold.

(i) Use HRR to show that

$$\chi(\mathcal{O}_X) = \frac{1}{24}\int_X c_1(X)c_2(X).$$

(This is the threefold equivalent of Noether's formula for surfaces obtaiend in class.) Furthermore, verify that $\int_X c_1(X)c_2(X)$ is indeed divisible by 24 when $X = \mathbb{P}^3$.

(ii) More generally, use HRR to obtain a formula for $\chi(X, \mathcal{O}_X(D))$ when D is a divisor over X.

(iii) If E is a rank 2 vector bundle over $\mathbb{P}^3$, use HRR to show that

$$\int_X c_1(E)c_2(E) \text{ is even.}$$

3. (*Practice with GRR.*) Let $L \to C$ be a line bundle over a smooth curve, and

$$X = \mathbb{P}(\mathcal{O}_C \oplus L) \to C$$

be the corresponding ruled surface. Let $\mathcal{O}_{\mathbb{P}}(1)$ denote the tautological bundle over the projective bundle $X = \mathbb{P}(\mathcal{O}_C \oplus L) \to C$.

(i) Explain that for $k \geq 0$ we have

$$\pi_\star \mathcal{O}_{\mathbb{P}}(k) = \oplus_{i=0}^k L^{-i}.$$

(ii) Use GRR to find the Chern character of $\pi_\star \mathcal{O}_{\mathbb{P}}(k)$ and verify that it is in agreement with the calculation in (i). That is, show that

$$\mathrm{ch}(\pi_\star \mathcal{O}_{\mathbb{P}}(k)) \cdot \mathrm{td}(C) = \pi_\star \left(\mathrm{ch}\ \mathcal{O}_{\mathbb{P}}(k) \cdot \mathrm{td}(X)\right).$$

4. (*K-theory of projective spaces.*) Show that

$$K(\mathbb{P}^n) = \mathbb{Z}[t]/(t-1)^{n+1}$$

where t corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

Hint: Use resolutions of coherent sheaves to show that t generates. To show that the relation $(t-1)^{n+1} = 0$ holds, you may need to work with the Koszul complex of the vector bundle $\oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1)$. To show that no other relations exist, reduce the degree to be less than $n+1$. Assuming

$$\sum_{i=0}^n a_i t^i = 0,$$

multiply by t^m for any m, and apply the Euler characteristic χ to both sides, regarding the answer as a polynomial in m. Examine this polynomial and show $a_i = 0$.

The last problem is *optional.* There are several hints to help you go through the argument by yourselves. If you get stuck, you can also consult Borel-Serre's Riemann-Roch paper.

5. (*Excision.*) If U is an open subset of an algebraic variety X, and $Y = X \setminus U$, then

$$K_\circ(Y) \to K_\circ(X) \to K_\circ(U) \to 0.$$

Strategy: We will prove this in 5 steps. Steps (i)-(iv) prove exactness on the right. They establish general extension properties of coherent sheaves (from U to X). Step (v) proves exactness in the middle.

Steps (i)-(iv) build on one another. The strategy can be summarized as follows:

– extension of subsheaves of a given sheaf that has already been extended;
– extension of globally generated sheaves;
– extensions of sheaves over affine varieties;
– extension of sheaves in the general case.

(i) If $\mathcal{A}$ is a coherent sheaf on X and $\mathcal{F}$ is a coherent sheaf on U, such that $\mathcal{F}$ is a subsheaf of $\mathcal{A}|_U$, then one can extend $\mathcal{F}$ to a coherent sheaf $\mathcal{F}'$ over X such that $\mathcal{F}'$ is a subsheaf of $\mathcal{A}$.

Hint: Consider $\mathcal{F}'(W)$ to be the set of sections of $\mathcal{A}(W)$ whose restrictions to $U \cap W$ are in $\mathcal{F}(U \cap W)$.

Now, the sheaf $\mathcal{A}$ may not exist, so we seek to remove it from our hypotheses.

(ii) If $\mathcal{F} \to U$ is a coherent sheaf over U generated by global sections, then there exists an extension $\mathcal{F}'$ of $\mathcal{F}$ to X.

Hint: Write $\mathcal{F} = \mathcal{O}_U^{\oplus r}/\mathcal{K}$ with $\mathcal{K}$ a subsheaf of $\mathcal{O}_U^{\oplus r}$. Use (i) to extend $\mathcal{K}$ to X, and thus also $\mathcal{F}$.

(iii) Assume X is affine. If $\mathcal{F} \to U$ is a coherent sheaf over U, then there exists an extension $\mathcal{F}'$ of $\mathcal{F}$ to X.

Hint: By (ii), it suffices to show that $\mathcal{F}$ is generated by global sections on U. To see this, take a principal affine open sets contained in $X_g \subset U$, and extend sections of $\mathcal{F}|_{X_g}$ to U possibly after multiplying by powers of g. This should follow using only techniques from Math 203b.

(iv) Show that the statement in (iii) holds without the assumption X affine. This implies that the excision sequence is exact on the right.

Hint: Show that $\mathcal{F}$ extends to *some* open V strictly containing U. This process will then end since open sets eventually stabilize. To find V, let W be an affine neighborhood of $x \in X \setminus U$, set $V = W \cup U$. Extend $\mathcal{F}|_{W \cap U}$ to a sheaf over W using (iii).

(v) Prove the excision sequence.

Hint: For exactness in the middle, you need to show that if $\mathcal{F}$ is extended to X in two different ways $\mathcal{F}'$ and $\mathcal{F}''$, then $[\mathcal{F}'] - [\mathcal{F}'']$ comes from $K(Y)$.

Assume first there exists a morphism $f : \mathcal{F}' \to \mathcal{F}''$ between the extensions. Look at the kernel and cokernel of f to conclude. To this end, you may wish to note that if $\mathcal{G}$ is a sheaf which is zero on U then $\mathcal{I}^n\mathcal{G} = 0$ where $\mathcal{I}$ is the ideal of Y in X (to see this, work affine locally). Construct filtrations of $\mathcal{G}$ whose succesive quotients are annihlated by $\mathcal{I}$, and conclude they come from Y.

In general, compare $\mathcal{F}'$ and $\mathcal{F}''$ to $\mathcal{F}' \oplus \mathcal{F}''$ to reduce to the case already analyzed.