**Math 203, Problem Set 2. Due Monday, April 22.**

**1.** (*Examples of flat morphisms.*)

(i) Give an example of a flat morphism $f : X \to T$ such that $f^{-1}(t)$ is irreducible for all $t \neq t_0$, while $f^{-1}(t_0)$ is not irreducible.

(ii) Give an example of a flat morphism $f : X \to T$ such that $f^{-1}(t)$ is reduced for all $t \neq t_0$, while $f^{-1}(t_0)$ is not reduced.

**2.** (*Flatness and pullbacks.*) Assume $f : X \to Y$ is a flat morphism. Show that exact sequences of coherent sheaves on $Y$ are still exact on $X$ after pull back.

**3.** (*Base-change and flatness.*)

(i) Show that if $M$ is an $A$-flat module, and $B$ is $A$-algebra, then $M \otimes_A B$ is $B$-flat.

(ii) Show that flatness $f : X \to Y$ is preserved under base-change by a morphism $g : Y' \to Y$.

**4.** (*Flatness and sections.*)

(i) Show that if $M$ is $B$-flat and $B$ is a flat $A$-algebra, then $M$ is $A$-flat.

(ii) Let $A$ be a $B$-algebra, and let $M$ be $A$-module. Show that $M$ is $B$-flat iff $M_{\mathfrak{p}}$ is $B$-flat for all primes $\mathfrak{p}$ in $A$.

(iii) Let $f : X \to Y$ be flat and $U \subset X$ and $V \subset Y$ are affine open sets with $U \subset f^{-1}(V)$. Show the flatness of the homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U).$$

Thus $f : \text{Spec} A \to \text{Spec } B$ is flat iff $f^\star : B \to A$ is flat.

**5.** (*Flatness over PID, DVR, dual numbers.*)

(i) Show that if $A$ is a PID, then $M$ if $A$-flat iff $M$ is torsion free.

(ii) Show that if $A$ is a DVR with local parameter $t$, then $M$ is $A$-flat iff $t$ is not a zero divisor on $M$.

(iii) Let $D = k[\epsilon]/(\epsilon^2)$. Show that $M$ is $D$-flat iff the multiplication by $\epsilon : M/\epsilon M \to \epsilon M$ is injective. This is important in deformation theory.

**6.** (*Example of a non-flat morphism.*) Let

$$X = \text{Spec } k[x, y, z, w]/(xz, yz, xw, yw)$$

be the union of the $(xy)$-plane and $(zw)$-plane in $\mathbb{A}^4$. Let

$$\pi : X \to \mathbb{A}^2$$

be given by

$$k[s, t] \to k[x, y, z, w]/(xz, yz, xw, yw), \quad s \to x + z, t \mapsto y + w.$$

Show that $\pi$ is not flat. You may wish to check the definition directly.

Geometrically, $X$ is a union of two planes intersecting at a point, and $\pi$ maps these planes isomorphically onto $\mathbb{A}^2$. This example should be contrasted with the criterion for flatness discussed in class when the base is a smooth curve.

The final question is *optional*.

**7.** (*Some deformation theory. The Hilbert "scheme".*)
Let

$$\mathcal{F} : (Sch) \to (Set)$$

be a contravariant functor between the category of schemes and the category of sets.

Let $X$ be an element of the set $\mathcal{F}(\text{Spec } k)$. The tangent space of $\mathcal{F}$ at $X$ is by definition the fiber over $X$ of the natural map

$$\mathcal{F}(\mathbb{D}) \to \mathcal{F}(\text{Spec } k).$$

Here $\mathbb{D} = \text{Spec } k[\epsilon]/(\epsilon^2)$.

(i) Let $X$ be a variety over $k$. Recall that in the first few lectures of Math 203b we have defined the functor of points

$$h_X : (Sch) \to (Set), \quad S \mapsto \text{Mor}(S, X).$$

Note that $h_X(\text{Spec } k) = X(k)$. Show that the tangent space of the functor $h_X$ at the point $p \in X$ coincides with the Zariski tangent space $T_{X,p}$. This should be immediate provided you solved an older homework in Math 203b.

(ii) The Hilbert functor

$$\mathcal{H} : (Sch) \to (Sets)$$

is defined by assigning to each scheme $S$, flat families

$$\pi : \mathcal{X} \to S, \quad \mathcal{X} \subset S \times \mathbb{P}^r$$

up to isomorphism of $S$-schemes. Using problem 3(ii), convince yourself that $\mathcal{H}$ is a contravariant functor.

(iii) If $X \subset \mathbb{P}^r$, then the tangent space of the Hilbert functor at $X$ is $\text{Hom}_{\mathbb{P}^r}(\mathcal{I}_X, \mathcal{O}_X)$. This tangent space parametrizes infinistesimal deformations of $X$ inside $\mathbb{P}^r$ i.e. $\mathbb{D}$-flat families

$$\mathcal{X} \to \mathbb{D}, \quad \mathcal{X} \subset \mathbb{P}^r \times \mathbb{D},$$

which restrict to $X$.

*Hint:* You should first treat the affine case, then glue. For the affine case, the problem can be rephrased as follows: given an ideal

$$\mathfrak{i} \subset A$$

find an ideal

$$\mathfrak{i}' \subset A' := A[\epsilon]/(\epsilon^2)$$

with $A'/\mathfrak{i}'$ flat over $D$, such that

$$A'/\mathfrak{i}' \otimes_D k = A/\mathfrak{i}.$$

Show these ideals are classified by $Hom_A(\mathfrak{i}, A/\mathfrak{i})$. To this end, the criterion of flatness in Problem 5 gives the exactness of

$$0 \to A/\mathfrak{i} \to^{\times\epsilon} A'/\mathfrak{i}' \to A/\mathfrak{i} \to 0.$$

Use this as a first row of a natural square diagram and diagram chase.

If you get suck, you can look this up in Hartshorne's deformation theory, but do write up your own solution.

(iii) In particular, if $X$ is smooth, the tangent space can be expressed as

$$\mathrm{Hom}_{\mathbb{P}^r}(\mathcal{I}_X, \mathcal{O}_X) = \mathrm{Hom}_X(\mathcal{I}_X/I_X^2, \mathcal{O}_X) = \mathrm{Hom}_X(N_{X/\mathbb{P}^r}^{\vee}, \mathcal{O}_X) = H^0(X, N_{X/\mathbb{P}^r}).$$

This is of course quite suggestive if you draw a picture.

It is a difficult result that the Hilbert functor is the functor of points of a certain scheme, called the Hilbert scheme. The above calculation therefore gives the tangent space to the Hilbert scheme, and it is useful to get an upper bound on the dimension.