Math 203 C - Problem Set 3, due Friday, May 3

1. (*Mayer-Vietoris.*) Show that if Y_1, Y_2 are two closed subsets of X, there is an exact sequence

$$A_k(Y_1 \cap Y_2) \to A_k(Y_1) \oplus A_k(Y_2) \to A_k(Y_1 \cup Y_2) \to 0.$$

2. (*Blowups.*) Compute the Chow groups of the blowup of $\mathbb{P}^2$ at n points.

3. (*Kunneth decomposition.*)

(i) Show that there are well-defined exterior products

$$A^k(X) \otimes A^\ell(Y) \to A^{k+\ell}(X \times Y)$$

which send

$$[Z] \otimes [W] \mapsto [Z \times W].$$

(ii) Show that if X admits an affine decomposition, then

$$\bigoplus_{k+\ell=m} A^k(X) \otimes A^\ell(Y) \to A^m(X \times Y)$$

is surjective. By contrast with algebraic topology, in general, this map is neither injective nor surjective.

4. (*Product of projective spaces.*)

(i) Compute the Chow groups of $X = \mathbb{P}^n \times \mathbb{P}^m$. By Problem 3, you should find generators given by products of linear spaces. To show that there are no relations you may wish to use projections via proper morphisms.

(ii) If Y is a hypersurface of bidegrees (d, e) in X, show that $[Y] = dh_1 + eh_2$ in $A^1(X)$, where h_1 and h_2 are the two hyperplane classes on X.

5. (*Counting fixed points.*) In this problem, we need to assume that there is an intersection product for $X = \mathbb{P}^n \times \mathbb{P}^m$. That is, the assignment

$$[V] \cdot [W] = [V \cap W]$$

if V, W are subvarieties intersecting transversally, makes

$$A^\star(X) = \oplus_k A^k(X)$$

into a ring. This fact holds true for any smooth X.

(i) Formulate a version of Bezout's theorem for $\mathbb{P}^n \times \mathbb{P}^m$ giving the number of intersection points of $(n+m)$ hypersurfaces of bidegrees (d_i, e_i) for $1 \le i \le n+m$.

(ii) Show that if $f : \mathbb{P}^n \to \mathbb{P}^n$ is a degree d morphism, then the class of the graph of f in $A^\star(\mathbb{P}^n \times \mathbb{P}^n)$ equals

$$[\Gamma_f] = \sum d^k \cdot h_1^k \cdot h_2^{n-k}.$$

Derive the class of the diagonal Δ in $A^\star(X)$ in terms of the generators h_1, h_2.

Hint: By Problem 4 you should be able to write

$$[\Gamma_f] = \sum_{i+j=n} c_i h_1^i h_2^j.$$

To find the coefficients c_{ij} you may wish to intersect with complementary classes.

(iii) Find the number of fixed points of a morphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree d.