Math 203, Problem Set 4. Due Wednesday, May 22.

1. (Tensor products.)

If $E, F$ are two rank 2 vector bundles, compute the Chern classes of $E \otimes F$ in terms of the Chern classes of $E, F$.

2. (Lines on the intersection of two quadrics.)

Let $X$ be the smooth intersection of two quadrics in $\mathbb{P}^4$. Show that the expected number of lines in $X$, counted with multiplicity, equals 16.

*Hint:* There are several ways of solving this problem. One way is via a Chern class calculation as we did in class for the 27 lines on the cubic surface.

You can also approach this problem by first finding a basis for the Chow group of the Grassmannian $G(1, \mathbb{P}^4)$ via a cellular decomposition. Consider the cycle of lines lying on a quadric. Express this cycle in terms of the basis you found above, possibly by intersecting with complementary subvarieties. Conclude.

*Remark:* In fact, just as for the cubic surface, the geometry can be made more precise. A smooth intersection of two quadrics in $\mathbb{P}^4$ is a del Pezzo surface of degree 4 (that is, $X$ is a smooth projective surface with $K_X^2$ ample and $K_X^2 = 4$.) It can be shown that all such del Pezzo surfaces be realized as a blowup of $\mathbb{P}^2$ at 5 general points. Assuming this description, can you figure out what the 16 lines are?

3. (Lines on the quintic threefold.)

Show that the expected number of lines on a quintic threefold in $\mathbb{P}^4$ is 2,875.

*Remark:* Clemens conjectured that the number of rational curves of a given degree on a general quintic threefold is finite. (Some smooth but non-generic quintic threefolds have infinite families of lines on them.)

Let $n_d$ be the expected number of degree $d$ rational curves on a quintic threefold. (A rigorous definition requires Gromov-Witten theory.) The genus 0 mirror theorem gives the formula for $n_d$. It takes the form

$$5 + \sum_d n_d \cdot d^3 \cdot \frac{q^d}{1 - q^d} = \frac{5}{1 + 5^5 x \cdot I_0(x)} \cdot \left( \frac{x \cdot dx}{q \cdot dq} \right)^3.$$ 

The variables $x$ and $q$ are related by the so-called mirror map. Specifically

$$q = \exp \left( \frac{I_1(x)}{I_0(x)} \right)$$
where 
\[ I_0(x) = \sum_n \frac{(5n)!}{(n!)^5} (-x)^n \]
\[ I_1(x) = I_0(x) \log(-x) + 5 \sum_n \frac{(5n)!}{(n!)^5} \left( \sum_{j=n+1}^{5n} \frac{1}{j} \right) \cdot (-x)^n. \]

The right hand side is explicit, so this provides an explicit formula for \( n_d \). In particular

\[ n_1 = 2,875, \quad n_2 = 609,250, \quad n_3 = 317,206,375, \quad n_4 = 242,467,530,000, \ldots \]

Needless to say, a proper explanation for the mysterious appearance of the hypergeometric series in the expression above is more involved.

4. (Chow groups of projective bundles.)

Let \( E \rightarrow X \) be a rank \( r \) vector bundle and let

\[ p : \mathbb{P}(E) \rightarrow X. \]

Let \( \zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \) be the Chern class of the tautological line bundle over \( \mathbb{P}(E) \). Show that

\[ \zeta^r + \zeta^{r-1} \cdot p^* c_1(E) + \ldots + p^* c_r(E) = 0 \]

in \( A_* (\mathbb{P}(E)) \).

**Hint:** The vector bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^* E \) has a natural section, hence using Whitney formula or an equivalent argument, its top Chern class must vanish

\[ c_r(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^* E) = 0. \]

5. (Optional: Conics intersecting 8 lines.)

Given 8 lines \( L_1, \ldots, L_8 \) in \( \mathbb{P}^3 \) show the expected number of plane conics in \( \mathbb{P}^3 \) that intersect all \( L_i \) equals 92.

**Hint:** Let \( X = (\mathbb{P}^3)^* \) denote the dual projective space parametrizing planes in \( \mathbb{P}^3 \) and let

\[ 0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_X \otimes \mathbb{C}^4 \rightarrow \mathcal{O}_X(1) \rightarrow 0 \]

denote the tautological sequence. Convince yourselves that the space of plane conics in \( \mathbb{P}^3 \) is

\[ \mathbb{H} = \mathbb{P}(\text{Sym}^2 \mathcal{S}^*) \rightarrow X. \]

This corresponds to the fact that one needs to pick a plane in \( \mathbb{P}^3 \), and for each plane, there is a \( \mathbb{P}^5 \)-worth of conics in that plane, corresponding to a quadratic equation up to scaling.
There are (at least) two ways to carry out the calculation, but if you wish you can follow the steps below:

(i) Using Problem 4, find the Chow of this projective bundle $\mathbb{H} \to X$. You should find two generators $\zeta$ and $\tau$ corresponding to the Chern classes of $\mathcal{O}_\mathbb{H}(1)$ and $\mathcal{O}_X(1)$. There should also be one relation which you should write explicitly.

(ii) Show that the divisor $\delta$ of conics intersecting a line is

$$\delta = \zeta + 2\tau.$$  

Further hint: By (i), write

$$\delta = a\zeta + b\tau.$$  

To show

$$a = 1, b = 2,$$

intersect $\delta$ with suitably chosen test curves. That is, consider a one-parameter family of conics

$$C_t \subset \mathbb{P}^3$$

parametrized by $t \in T$. Such a family determines a curve in the space of conics

$$T \to \mathbb{H}, \quad t \mapsto [C_t].$$

Compute $T \cdot \delta$.

For instance, you can intersect with the following one-parameter families:

- a one-parameter family of conics $C_t, t \in \mathbb{P}^1$ that are contained in a general plane $H$ in $\mathbb{P}^3$. Show that

$$T \cdot \delta = 1, \quad T \cdot \zeta = 1, \quad T \cdot \tau = 0.$$  

- a one-parameter family of conics of the form

$$C_t = H_t \cap Q$$

where $H_t, t \in \mathbb{P}^3$ is a varying plane in $\mathbb{P}^3$, and $Q$ is a fixed quadric. Show that

$$T \cdot \delta = 2, \quad T \cdot \zeta = 0, \quad T \cdot \tau = 1.$$  

You will need to compute intersections with both families since you need to find two coefficients $a, b$.

(iii) Show that $\delta^8 = 92$. You may wish to begin observing that

$$\zeta^5 \tau^3 = 1 \text{ (why?)},$$

and use Problem 4 to compute that

$$\zeta^6 \tau^2 = -4, \quad \zeta^7 \tau = 6, \quad \zeta^8 = -4.$$