Math 203, Problem Set 4. Due Wednesday, May 22.

1. (*Tensor products.*)

If E, F are two rank 2 vector bundles, compute the Chern classes of $E \otimes F$ in terms of the Chern classes of E, F.

2. (*Lines on the intersection of two quadrics.*)

Let X be the smooth intersection of two quadrics in $\mathbb{P}^4$. Show that the expected number of lines in X, counted with multiplicity, equals 16.

Hint: There are several ways of solving this problem. One way is via a Chern class calculation as we did in class for the 27 lines on the cubic surface.

You can also approach this problem by first finding a basis for the Chow group of the Grassmannian $G(1, \mathbb{P}^4)$ via a cellular decomposition. Consider the cycle of lines lying on a quadric. Express this cycle in terms of the basis you found above, possibly by intersecting with complementary subvarieties. Conclude.

Remark: In fact, just as for the cubic surface, the geometry can be made more precise. A smooth intersection of two quadrics in $\mathbb{P}^4$ is a *del Pezzo surface* of degree 4 (that is, X is a smooth projective surface with $K_X^\vee$ ample and $K_X^2 = 4$.) It can be shown that all such del Pezzo surfaces be realized as a blowup of $\mathbb{P}^2$ at 5 general points. Assuming this description, can you figure out what the 16 lines are?

3. (*Lines on the quintic threefold.*)

Show that the expected number of lines on a quintic threefold in $\mathbb{P}^4$ is $2,875$.

Remark: Clemens conjectured that the number of rational curves of a given degree on a *general* quintic threefold is finite. (Some smooth but non-generic quintic threefolds have infinite families of lines on them.)

Let n_d be the expected number of degree d rational curves on a quintic threefold. (A rigorous definition requires Gromov-Witten theory.) The genus 0 *mirror theorem* gives the formula for n_d. It takes the form

$$5 + \sum_d n_d \cdot d^3 \cdot \frac{q^d}{1 - q^d} = \frac{5}{1 + 5^5 x \cdot I_0(x)} \cdot \left(\frac{x}{q} \cdot \frac{dx}{dq} \right)^3 .$$

The variables x and q are related by the so-called *mirror map*. Specifically

$$q = \exp\left(\frac{I_1(x)}{I_0(x)} \right)$$

where

$$I_0(x) = \sum_n \frac{(5n)!}{(n!)^5}(-x)^n$$

$$I_1(x) = I_0(x)\log(-x) + 5\sum \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)\cdot(-x)^n.$$

The right hand side is explicit, so this provides an explicit formula for n_d. In particular

$$n_1 = 2,875, \;\; n_2 = 609,250, \;\; n_3 = 317,206,375, \;\; n_4 = 242,467,530,000\ldots.$$

Needless to say, a proper explanation for the mysterious appearance of the hypergeometric series in the expression above is more involved.

4. (*Chow groups of projective bundles.*)

Let $E \to X$ be a rank r vector bundle and let

$$p : \mathbb{P}(E) \to X.$$

Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ be the Chern class of the tautological line bundle over $\mathbb{P}(E)$. Show that

$$\zeta^r + \zeta^{r-1}\cdot p^\star c_1(E) + \ldots + p^\star c_r(E) = 0$$

in $A_\star(\mathbb{P}(E))$.

Hint: The vector bundle $\mathcal{O}_{\mathbb{P}(E)}(1)\otimes p^\star E$ has a natural section, hence using Whitney formula or an equivalent arguement, its top Chern class must vanish

$$c_r(\mathcal{O}_{\mathbb{P}(E)}(1)\otimes p^\star E) = 0.$$

5. (*Optional: Conics intersecting* 8 *lines.*)

Given 8 lines $L_1, \ldots L_8$ in $\mathbb{P}^3$ show the expected number of plane conics in $\mathbb{P}^3$ that intersect all L_i equals 92.

Hint: Let $X = \left(\mathbb{P}^3\right)^\star$ denote the dual projective space parametrizing planes in $\mathbb{P}^3$ and let

$$0 \to \mathcal{S} \to \mathcal{O}_X \otimes \mathbb{C}^4 \to \mathcal{O}_X(1) \to 0$$

denote the tautological sequence. Convince yourselves that the space of plane conics in $\mathbb{P}^3$ is

$$\mathbb{H} = \mathbb{P}(\mathrm{Sym}^2\mathcal{S}^\vee) \to X.$$

This corresponds to the fact that one needs to pick a plane in $\mathbb{P}^3$, and for each plane, there is a $\mathbb{P}^5$-worth of conics in that plane, corresponding to a quadratic equation up to scaling.

There are (at least) two ways to carry out the calculation, but if you wish you can follow the steps below:

(i) Using Problem 4, find the Chow of this projective bundle $\mathbb{H} \to X$. You should find two generators ζ and τ corresponding to the Chern classes of $\mathcal{O}_{\mathbb{H}}(1)$ and $\mathcal{O}_X(1)$. There should also be one relation which you should write explicitly.

(ii) Show that the divisor δ of conics intersecting a line is

$$\delta = \zeta + 2\tau.$$

Further hint: By (i), write

$$\delta = a\zeta + b\tau.$$

To show

$$a = 1, b = 2,$$

intersect δ with suitably chosen *test curves* . That is, consider a one-parameter family of conics

$$C_t \subset \mathbb{P}^3$$

parametrized by $t \in T$. Such a family determines a curve in the space of conics

$$T \to \mathbb{H}, \quad t \mapsto [C_t].$$

Compute $T \cdot \delta$.

For instance, you can intersect with the following one-parameter families:

– a one-parameter family of conics $C_t, t \in \mathbb{P}^1$ that are contained in a general plane H in $\mathbb{P}^3$. Show that

$$T \cdot \delta = 1, \;\; T \cdot \zeta = 1, \quad T \cdot \tau = 0.$$

– a one-parameter family of conics of the form

$$C_t = H_t \cap Q$$

where $H_t, t \in \mathbb{P}^1$ is a varying plane in $\mathbb{P}^3$, and Q is a fixed quadric. Show that

$$T \cdot \delta = 2, \;\; T \cdot \zeta = 0, \quad T \cdot \tau = 1.$$

You will need to compute intersections with both families since you need to find two coefficients a, b.

(iii) Show that $\delta^8 = 92$. You may wish to begin observing that

$$\zeta^5 \tau^3 = 1 \text{ (why?)},$$

and use Problem 4 to compute that

$$\zeta^6 \tau^2 = -4, \;\; \zeta^7 \tau = 6, \;\; \zeta^8 = -4.$$