Math 203, Problem Set 2. Due Monday, January 29.

1. (*Sheaves of modules.*) Let X be a scheme. A sheaf $\mathcal{F} \to X$ is said to be a sheaf of $\mathcal{O}_X$-modules provided that for all open sets $U \subset X$, $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ in a fashion compatible with restrictions.

(i) Make this definition precise.

(ii) Now assume $X = \text{Spec } A$, and let M be an A-module. Show that the assignment

$$X_f \to \mathcal{F}(X_f) = M_f$$

defines a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.

(iii) Show furthermore that the stalks of this sheaf are

$$\mathcal{F}_{\mathfrak{p}} = M_{\mathfrak{p}}.$$

We will write $\widetilde{M}$ for this sheaf.

Remark: This is very similar to a proof given in class. Give as many details as you deem necessary.

2. Consider the arithmetic surface $X = \text{Spec } \mathbb{Z}[x] = \mathbb{A}^1_{\mathbb{Z}}$. Let $\mathfrak{p} = (2, x)$ be a point in X. Show that regular functions over $X \setminus \{\mathfrak{p}\}$ extend to X.

Remark: This should be paralleled with the statement for $\mathbb{A}^2_{\mathbb{C}}$ proved last quarter. Just as in the proof given last quarter, cover X by two suitable distinguished open sets X_f and X_g.

3. Let $X = \mathbb{P}^1_{\mathbb{Z}}$. Show that the set of morphisms $\text{Spec } \mathbb{Z} \to X$ is $\mathbb{Q} \cup \{\infty\}$.

Hint: Cover X by the two standard affine sets $\text{Spec } \mathbb{Z}[x]$ and $\text{Spec } \mathbb{Z}[1/x]$ and consider their pre-images U and V in $\text{Spec } \mathbb{Z}$. The sets U, V are open in $\text{Spec } \mathbb{Z}$. What are the open sets in $\text{Spec } \mathbb{Z}$?

4. Let X be a scheme. For each $x \in X$, write $k(x)$ for the residue field.

(i) Let K be a field. Show that to give a morphism $\text{Spec}(K) \to X$ is the same as giving a point of X and an inclusion $k(x) \to K$.

(ii) Show that there is a natural morphism $\text{Spec } \mathcal{O}_{X,x} \to X$. Show that the image can be identified with the intersection of all open subsets containing x.

5. A scheme X is locally of finite type over a field k if it can be covered by affine open sets $U_i = \text{Spec } A_i$ with A_i a finitely generated k-algebra. Let X be a scheme locally of finite type over k.

(i) Show that x is a closed point iff the extension $k \subset k(x)$ is finite.

Hint: In one direction, recall the following version of the Nullstellensatz: for a maximal ideal $\mathfrak{m}$ in a finitely generated k-algebra A, $A/\mathfrak{m}$ is finite over k.

(ii) In particular, show that if k is algebraically closed, then the set of closed points of X can be identified with

$$X(k) = \mathrm{Hom}(\mathrm{Spec}\ k, X).$$

(iii) Show that if $x \in U \cap V$ for two affine open sets $U, V \subset X$, then x is closed in U iff x is closed in V.

(iv) Show that the set of closed points is dense in X (in fact, in every closed subset of X).

(v) On the other hand, give an example of a scheme whose closed points are not dense.

6. Let X be a prevariety over a field k, and let $p \in X$ be a closed point of X. Let

$$D = \mathrm{Spec}\ k[\epsilon]/(\epsilon^2)$$

be the *double point*. Show that the tangent space $T_{X,p}$ to X at p can be identified with the set of morphisms $D \to X$ that map the unique point of D to p.

7. (*Reduced schemes.*) Show that for a scheme X the following are equivalent:

(i) X is *reduced* i.e. for all $U \subset X$ open, $\mathcal{O}_X(U)$ has no nilpotents;

(ii) for all $p \in X$, the local rings $\mathcal{O}_{X,p}$ have no nilpotents.