Math 203, Problem Set 3. Due Wednesday, February 7.

Hand in either Problem 1 or Problem 2.

1. (*Morphisms locally of finite type.*) Show that being a morphism locally of finite type is affine local on the target.

That is, show that if $f : X \to Y$ is a morphism locally of finite type, then for all affine open $V \subset Y$, $V = \text{Spec } B$, the preimage $f^{-1}(V)$ can be covered by affine open sets $U_j = \text{Spec } A_j$ with A_j a finitely generated B-algebra.

2. (*Finite morphisms.*) Show that being a finite morphism is affine local on the target.

That is, show that if $f : X \to Y$ is finite, then for all affine open $V \subset Y$, $V = \text{Spec } B$, the preimage $f^{-1}(V)$ is affine Spec (A) with A a finite B-module.

Also show that any closed immersion is finite.

Hand in Problem 3.

3. (*Quasi-finite morphisms.*) A morphism $f : X \to Y$ if quasi-finite if its fibers are finite sets. (Often, one also asks that f be of finite type for base change property to hold).

Clearly, being quasi-finite is local on the target. In this problem, show that f finite implies f quasi-finite but the converse is false.

Hand in one of Problem 4, 5 or 6.

4. (*Separated morphisms.*) Show that the property of a morphism to be separated is affine-local on the target.

5. (*Proper morphisms.*) Show that the property of a morphism to be proper is affine-local on the target.

6. (*Quasi-compact morphisms.*) Show that the property of a morphism to be quasi-compact is local on the target.

Hand in both Problem 7 and 8.

7. (*Separated morphisms.*) Let X be separated over S, where S is affine. Show that if U, V are affine open in X then $U \cap V$ is affine open as well.

Hint: Closed immersions, as defined in class, are automatically affine morphisms.

8. (*Affine morphisms – this is trickier than the rest.*) Show that the property of a morphism to be affine is affine local on the target.

We have seen in class that this reduces to the following statement: if X is any scheme with $A = \mathcal{O}_X(X)$, let $f_1, \ldots, f_n$ be elements which generated the unit ideal in A. Show that

$$X \text{ is affine} \iff X_{f_k} \text{ is affine for all } k.$$

One direction should be clear. For the converse proceed as follows:

(i) Assume $X_{f_k} = \text{Spec } A_k$ for some A-algebra A_k. We claim that

$$A_k = A_{f_k}.$$

To prove this fact, consider $X_{ij} = X_i \cap X_j$. Show that X_{ij} is affine Spec A_{ij} where $A_{ij} = (A_i)_{f_j}$. Next, consider the exact sequence

$$0 \to \mathcal{O}_X(X) \to \oplus_{i=1}^n \mathcal{O}_X(X_i) \to \oplus_{ij} \mathcal{O}_X(X_{ij}).$$

Explain that

$$0 \to A \to \oplus A_i \to \oplus A_{ij}.$$

Localize at f_k and conclude.

(ii) Consider the natural morphism $X \to$ Spec A. Recall that being an isomorphism is local on the target. Use this fact and part (i) to conclude that

$$X \simeq \text{Spec } A.$$