Math 203, Problem Set 5. Due Friday, February 23.

1. (*Projective bundles.*) Let $E \to X$ be a vector bundle over a variety X. Consider the projectivization

$$\pi : \mathbb{P}(E) \to X,$$

and recall the line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ with the natural morphism

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(-1) \to \pi^\star E.$$

(i) Show that a morphism $f : S \to \mathbb{P}(E)$ yields a morphism $\eta = \pi \circ f : S \to X$ together with a line bundle $\mathcal{L} \to S$ and a morphism of vector bundles

$$0 \to \mathcal{L} \to \eta^\star E.$$

The converse is also true, and you should convince yourself this is the case by working locally (or read Hartshorne II.7.12).

(ii) Calculate the canonical line bundle $K_{\mathbb{P}(E)}$ in terms of the canonical bundle of X, the determinant of E and twisting line bundles.

Hint: You should convince yourself that the generalized Euler sequence is exact by working locally:

$$0 \to \Omega_{\mathbb{P}(E)/X} \to \mathcal{O}_{\mathbb{P}E}(-1) \otimes \pi^\star E^\vee \to \mathcal{O} \to 0.$$

(iii) Consider the *Hirzebruch surface*

$$\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(a)).$$

What is the canonical bundle of $\mathbb{F}_a$?

(iv) Let $\mathbf{F}$ denote the *flag variety* parametrizing pairs of subspaces

$$V_1 \subset V_2 \subset \mathbb{C}^3$$

where $\dim V_1 = 1, \dim V_2 = 2$. This describes $\mathbf{F}$ set theoretically, but you can give $\mathbf{F}$ a scheme structure in such a fashion that there is a canonical sequence of vector bundles

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathbb{C}^3 \otimes \mathcal{O}_{\mathbf{F}},$$

where the fibers of $\mathcal{V}_1, \mathcal{V}_2$ over the point $V_1 \subset V_2 \subset \mathbb{C}^3$ are V_1 and V_2 respectively. In fact, you can construct $\mathbf{F}$ as a projective bundle over the (dual) projective space. There are several statements to be checked here, and part (i) may help; just include as many details as you deem necessary.

In any case, calculate the canonical bundle of $\mathbf{F}$ in terms of $\mathcal{V}_1, \mathcal{V}_2$.

2. (*Adjunction formula.*) Let X be a smooth variety, and let $Y \subset X$ be a smooth hypersurface. Show that $N_{Y/X} \simeq \mathcal{O}_X(Y)|_Y$. Conclude from the normal sequence that

$$K_Y = K_X \otimes \mathcal{O}_X(Y)|_Y.$$

3. (*Complete intersections.*) Let X be a complete intersection of hypersurfaces of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$.

(i) Show that $K_X = \mathcal{O}_X(\sum d_i - n - 1)$.

Fact: We will see later that if X is a complete intersection then the restriction

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \to H^0(X, \mathcal{O}_X(\ell))$$

is surjective for ℓ. A direct argument is possible, but the shortest proofs use cohomology. In any case, we will assume this here. A normal variety X with this property is said to be *projectively normal.*

(ii) The geometric genus p_g of X is defined as the number of sections of K_X. Show that if X is a degree d hypersurface in $\mathbb{P}^n$ then $p_g(X) = \binom{d-1}{n}$.

In particular if X is a smooth degree d curve in $\mathbb{P}^2$, then the geometric and arithmetic genus agree

$$p_g(X) = p_a(X) = \frac{(d-1)(d-2)}{2}.$$

Recall that the arithmetic genus was defined in the last homework of Math 203a.

(iii) A $K3$ surface is a smooth surface with trivial canonical bundle. (In addition, one requires that the irregularity be zero, see below, but we ignore this here.) K3s are examples of Calabi-Yau manifolds in dimension 2 and play a special role in the classification of surfaces. They are named after Kähler, Kummer and Kodaira and the mountain $K3$.

Classify all $K3$ surfaces which are complete intersections. What are their degrees in projective space?

Furthermore, if $X \subset \mathbb{P}^n$ is such a $K3$ surface on your list, and C is a smooth hyperplane section of X:

$$C = X \cap H$$

where H is a hyperplane in $\mathbb{P}^n$, compute the arithmetic genus of C.

4. (*Plurigenera, Hodge numbers, irregularity.*) Define the *plurigenera* of a smooth projective variety X to be

$$p_n(X) = \dim_k H^0(X, K_X^{\otimes n}).$$

This recovers the geometric genus when $n = 1$.

Define the *Hodge numbers*

$$h^{q,0}(X) = \dim H^0(X, \Omega^q).$$

The first Hodge number $q(X) = h^{1,0}(X)$ is called the *irregularity* of X.

Show that birational smooth varieties X and X' have the same pluri-genera and the same Hodge numbers $h^{q,0}$.

Hint: This follows by the methods of Hartshorne, Chapter II.8.19.

5. (*Product schemes, Kodaira dimension, geometric and arithmetic genera.*)

(i) Let X, Y be smooth varieties. Show that

$$\Omega_{X\times Y} = \mathrm{pr}_X^\star \Omega_X \oplus \mathrm{pr}_Y^\star \Omega_Y.$$

What is the corresponding statement for the canonical bundles?

(ii) Consider E a smooth cubic in $\mathbb{P}^2$. Show that for the surface $A = E \times E$, the arithmetic and geometric genus are -1 and 1, hence not equal.

Fact: It can be shown that the plurigenera (of smooth varieties) grow in such a fashion that

$$p_n(X)/n^k \text{ is bounded,}$$

for some constant

$$\mathrm{kod}(X) = k \in \{-\infty, 0, 1, \ldots, \dim X\}.$$

The constant k is called the *Kodaira dimension* of X. If kod $(X) = \dim X$ then X is said to be of general type.

(iii) Show that

$$\mathrm{kod}(X \times Y) = \mathrm{kod}\ (X) + \mathrm{kod}\ (Y).$$

(iv) If X is a smooth projective variety which is rational, show that kod $(X) = -\infty$ and the Hodge numbers $h^{q,0} = 0$.

(v) For arbitrary dimensions d, construct smooth projective varieties with Kodaira dimensions $-\infty, 0, 1, \ldots, d$.

Hint: In dimension $d = 1$, calculate the plurigenera of cubics and quartics in $\mathbb{P}^2$ using the methods of Problem 3. Then take products.

6. (*Blowups.*) Let $\pi : \tilde{X} \to X$ be the blowup of a smooth variety X of dimension d at a point p. Recall the exceptional divisor

$$E \simeq \mathbb{P}^{d-1}.$$

It is proven in Hartshorne, Theorem 8.24, that the normal bundle of E in $\tilde{X}$ equals $\mathcal{O}_E(-1)$. Use this fact to show that

$$K_{\tilde{X}} = \pi^\star K_X \otimes \mathcal{O}((d-1)E).$$

Hint: Consider $M = K_{\tilde{X}} \otimes \pi^\star K_X^\vee$. Show that M restricts trivially to $\tilde{X} \setminus E$. Show that this implies M is of the form $\mathcal{O}_{\tilde{X}}(qE)$. To confirm that $q = d - 1$, use adjunction formula for $(E, \tilde{X})$.