Math 203, Problem Set 5. Due Friday, February 23.

1. (Projective bundles.) Let $E \to X$ be a vector bundle over a variety $X$. Consider the projectivization

$$\pi : \mathbb{P}(E) \to X,$$

and recall the line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ with the natural morphism

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(-1) \to \pi^* E.$$

(i) Show that a morphism $f : S \to \mathbb{P}(E)$ yields a morphism $\eta = \pi \circ f : S \to X$ together with a line bundle $\mathcal{L} \to S$ and a morphism of vector bundles

$$0 \to \mathcal{L} \to \eta^* E.$$

The converse is also true, and you should convince yourself this is the case by working locally (or read Hartshorne II.7.12).

(ii) Calculate the canonical line bundle $K_{\mathbb{P}(E)}$ in terms of the canonical bundle of $X$ and the determinant of $E$.

Hint: You should convince yourself that the generalized Euler sequence is exact by working locally:

$$0 \to \Omega_{\mathbb{P}(E)/X} \to \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* E^\vee \to \mathcal{O} \to 0.$$

(iii) Consider the Hirzebruch surface

$$\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}1} + \mathcal{O}_{\mathbb{P}1}(a)).$$

What is the canonical bundle of $\mathbb{F}_a$?

(iv) Let $\mathbf{F}$ denote the flag variety parametrizing pairs of subspaces

$$V_1 \subset V_2 \subset \mathbb{C}^3$$

where $\dim V_1 = 1, \dim V_2 = 2$. This describes $\mathbf{F}$ set theoretically, but you can give $\mathbf{F}$ a scheme structure in such a fashion that there is a canonical sequence of vector bundles

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathbb{C}^3 \otimes \mathcal{O}_{\mathbf{F}},$$

where the fibers of $\mathcal{V}_1, \mathcal{V}_2$ over the point $V_1 \subset V_2 \subset \mathbb{C}^3$ are $V_1$ and $V_2$ respectively. In fact, you can construct $\mathbf{F}$ as a projective bundle over the (dual) projective space. There are several statements to be checked here, and part (i) may help; just include as many details as you deem necessary.

In any case, calculate the canonical bundle of $\mathbf{F}$ in terms of $\mathcal{V}_1, \mathcal{V}_2$.

2. (Adjunction formula.) Let $X$ be a smooth variety, and let $Y \subset X$ be a smooth hypersurface. Show that $N_{Y/X} \simeq \mathcal{O}_X(Y)|_Y$. Conclude from the normal sequence that

$$K_Y = K_X \otimes \mathcal{O}_X(Y)|_Y.$$
3. (Complete intersections.) Let $X$ be a complete intersection of hypersurfaces of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$.

(i) Show that $K_X = \mathcal{O}_X(\sum d_i - n - 1)$.

Fact: We will see later that if $X$ is a complete intersection then the restriction

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \to H^0(X, \mathcal{O}_X(\ell))$$

is surjective for $\ell$. A direct argument is possible, but the shortest proofs use cohomology. In any case, we will assume this here. A normal variety $X$ with this property is said to be projectively normal.

(ii) The geometric genus $p_g$ of $X$ is defined as the number of sections of $K_X$. Show that if $X$ is a degree $d$ hypersurface in $\mathbb{P}^n$ then $p_g(X) = (d-1)$.

In particular if $X$ is a smooth degree $d$ curve in $\mathbb{P}^2$, then the geometric and arithmetic genus agree

$$p_g(X) = p_a(X) = \frac{(d-1)(d-2)}{2}.$$ 

Recall that the arithmetic genus was defined in the last homework of Math 203a.

(iii) A $K3$ surface is a smooth surface with trivial canonical bundle. (In addition, one requires that the irregularity be zero, see below, but we ignore this here.) $K3$s are examples of Calabi-Yau manifolds in dimension 2 and play a special role in the classification of surfaces. They are named after Kähler, Kummer and Kodaira and the mountain $K3$.

Classify all $K3$ surfaces which are complete intersections. What are their degrees in projective space?

Furthermore, if $X \subset \mathbb{P}^n$ is such a $K3$ surface on your list, and $C$ is a smooth hyperplane section of $X$:

$$C = X \cap H$$

where $H$ is a hyperplane in $\mathbb{P}^n$, compute the arithmetic genus of $C$.

4. (Plurigenera, Hodge numbers, irregularity.) Define the plurigenera of a smooth projective variety $X$ to be

$$p_n(X) = \dim_k H^0(X, K_X^{\otimes n}).$$

This recovers the geometric genus when $n = 1$.

Define the Hodge numbers

$$h^{q,0}(X) = \dim H^0(X, \Omega^q).$$

The first Hodge number $q(X) = h^{1,0}(X)$ is called the irregularity of $X$. 
Show that birational smooth varieties $X$ and $X'$ have the same pluri-genera and the same Hodge numbers $h^{q,0}$.

**Hint:** This follows by the methods of Hartshorne, Chapter II.8.19.

5. *(Product schemes, Kodaira dimension, geometric and arithmetic genera.)*

   (i) Let $X, Y$ be smooth varieties. Show that

   $$\Omega_{X \times Y} = \text{pr}_X^* \Omega_X \oplus \text{pr}_Y^* \Omega_Y.$$ 

   What is the corresponding statement for the canonical bundles?

   (ii) Consider $E$ a smooth cubic in $\mathbb{P}^2$. Show that for the surface $A = E \times E$, the arithmetic and geometric genus are $-1$ and $1$, hence not equal.

   **Fact:** It can be shown that the plurigenera (of smooth varieties) grow in such a fashion that $p_n(X)/n^k$ is bounded, for some constant $k = \kod(X) = \{ -\infty, 0, 1, \ldots, \dim X \}$.

   The constant $k$ is called the *Kodaira dimension* of $X$. If $\kod(X) = \dim X$ then $X$ is said to be of general type.

   (iii) Show that

   $$\kod(X \times Y) = \kod(X) + \kod(Y).$$

   (iv) If $X$ is a smooth projective variety which is rational, show that $\kod(X) = -\infty$ and the Hodge numbers $h^{q,0} = 0$.

   (v) For arbitrary dimensions $d$, construct smooth projective varieties with Kodaira dimensions $-\infty, 0, 1, \ldots, d$.

   **Hint:** In dimension $d = 1$, calculate the plurigenera of cubics and quartics in $\mathbb{P}^2$ using the methods of Problem 3. Then take products.

6. *(Blowups.)* Let $\pi : \tilde{X} \to X$ be the blowup of a smooth variety $X$ of dimension $d$ at a point $p$. Recall the exceptional divisor

   $$E \simeq \mathbb{P}^{d-1}.$$ 

   It is proven in Hartshorne, Theorem 8.24, that the normal bundle of $E$ in $\tilde{X}$ equals $O_E(-1)$. Use this fact to show that

   $$K_{\tilde{X}} = \pi^* K_X \otimes O((d-1)E).$$

   **Hint:** Consider $M = K_{\tilde{X}} \otimes \pi^* K_{\tilde{X}}^\vee$. Show that $M$ restricts trivially to $\tilde{X} \setminus E$. Show that this implies $M$ is of the form $O_{\tilde{X}}(qE)$. To confirm that $q = d - 1$, use adjunction formula for $(E, \tilde{X})$. 