
1. (Affine morphisms.) Let \( f : X \to Y \) be an affine morphism between two Noetherian separated schemes, and let \( F \to X \) be a quasicoherent sheaf. Show that
\[
H^i(X, F) = H^i(Y, f_*F).
\]
*Hint:* Consider the affine open cover \( \mathcal{U} \) of \( Y \), and the associated cover \( f^{-1}(\mathcal{U}) \) for \( X \).

2. (Hilbert polynomials and arithmetic genus.) Let \( F \to X \) be a coherent sheaf over a projective variety \( X \subset \mathbb{P}^r \).
   (i) Show that there exists a polynomial \( \chi_F \) such that
   \[
   \chi(F(n)) = \chi_F(n).
   \]
   *Hint:* First reduce to the case \( X = \mathbb{P}^r \) by considering \( \iota_* F \). Over \( \mathbb{P}^r \), you may wish to argue by induction on \( r \), using the exact sequence
   \[
   0 \to F(-H) \to F \to F|_H \to 0
   \]
   for a suitable hyperplane \( H \subset \mathbb{P}^r \). The only issue is exactness on the left. You can construct a suitable hyperplane working over affine patches and using a bit of commutative algebra (associated points).
   (ii) In particular, if \( F = \mathcal{O}_X \) this recovers the Hilbert polynomial we introduced in Math 203a.
   (iii) Calculate the Hilbert polynomial of \( \mathcal{O}_{\mathbb{P}^r}(m) \).
   (iv) Using (ii), show that the arithmetic genus of \( X \) is defined as
   \[
   p_a(X) = (-1)^{\dim X - 1}(\chi(\mathcal{O}_X) - 1).
   \]
   In particular, if \( X \) is an irreducible projective curve then
   \[
   p_a(X) = \dim H^1(X, \mathcal{O}_X).
   \]
   This definition is independent of the projective embedding.

3. (Complete intersections.) Let \( X \subset \mathbb{P}^r \) be a smooth complete intersection in projective space with \( \dim X = d \). Show that
   (i) for all \( \ell \), the map
   \[
   H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) \to H^0(X, \mathcal{O}_X(\ell))
   \]
   is surjective. This was stated as a fact in a previous problem set, but now you have the tools to construct a proof.
   (ii) In particular, setting \( \ell = 0 \), \( X \) is connected.
   (iii) Show that the intermediate cohomology of the twisting line bundles vanishes
   \[
   H^i(X, \mathcal{O}_X(\ell)) = 0, \ 0 < i < d, \text{ for all } \ell.
   \]
Remark: A sheaf \( \mathcal{F} \) is said to be ACM (arithmetically Cohen-Macaulay) if \( \mathcal{F}(\ell) \) has no intermediate cohomology for all \( \ell \). Over complete intersections, \( \mathcal{O}_X \) is ACM.

(iv) Using problem 2, show that the arithmetic genus \( p_a(X) = \dim H^d(X, \mathcal{O}_X) \).

4. (Hodge numbers.) For a smooth projective variety over \( k \), set \( H^{p,q}(X) = H^q(X, \Omega^p_X) \) and define the Hodge numbers

\[
h^{p,q}(X) = \dim_k H^{p,q}(X).
\]

Show that for projective space \( h^{p,q}(\mathbb{P}^n) = 0 \) if \( p \neq q \) and \( h^{p,q}(\mathbb{P}^n) = 1 \) if \( p = q \).

Remark: If \( X \) is a smooth complex projective variety, then the cohomology of \( X \) splits as

\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).
\]

This is called the Hodge decomposition. The Hodge numbers \( h^{p,q} \) satisfy the symmetries

\[
h^{p,q} = h^{q,p} \quad \text{and} \quad h^{p,q} = h^{d-p,d-q}.
\]

The latter symmetry follows from Serre duality. The Betti numbers of \( X \) can be calculated from the Hodge numbers

\[
b_k(X) = \sum_{p+q=k} h^{p,q}.
\]

The Hodge numbers (arranged in a rotated square) form the Hodge diamond of \( X \).

Hint: You may wish to start with the Euler sequence

\[
0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n} (-1) \otimes \mathbb{C}^{n+1} \to \mathcal{O}.
\]

It is important to show that if

\[
0 \to E \to F \to \mathcal{O} \to 0
\]

is an exact sequence of vector bundles then you also have exactness of

\[
0 \to \Lambda^p E \to \Lambda^p F \to \Lambda^{p-1} E \to 0.
\]

5. (The Picard group.) Let \( X \) be a variety. Show that the Pic(\( X \)) can be identified with \( H^1(X, \mathcal{O}_X^*) \) where \( \mathcal{O}_X^* \) is the sheaf of nowhere-zero regular functions.

Hint: Start with trivializations \( \psi_i \) of a line bundle \( L|_{U_i} \to \mathcal{O}_{U_i} \) over open sets \( U_i \), and consider \( \psi_{ij} = \psi_i \circ \psi_j^{-1} \) over \( U_{ij} \). Show that \( \psi_{ij} \) satisfy the cocycle condition and thus define an element in \( H^1(X, \mathcal{O}_X^*) \).

6. (On finiteness of cohomology.) Let \( X = \mathbb{A}_k^2 \setminus \{(0,0)\} \). Show that \( \dim_k H^1(X, \mathcal{O}_X) \) is infinite.

Hint: Use the Cech cover with two open sets \( U = \{x \neq 0\} \) and \( V = \{y \neq 0\} \).