Math 203, Problem Set 7. Due Friday, March 16.

You should be able to solve Problems $1-3$ *on Friday, March 9. Problem* 4 *requires the material from Wednesday's lecture.*

1. (*Riemann-Roch in higher rank.*) Show that if X is a smooth projective curve and $E \to X$ is a rank r vector bundle, then

$$\chi(X, E) = r\chi(X, \mathcal{O}_X) + \deg\,(\Lambda^r E).$$

(i) Show first that it suffices to assume E admits a section.

Hint: Specifically, show that the above Riemann-Roch formula holds for E if and only if it also holds for $E \otimes \mathcal{O}(p)$. The same is true for $E \otimes L$ where L is any line bundle. Show that if L is suitably chosen, then $E \otimes L$ has a section.

(ii) If E has a section, let F be the sheaf given by

$$0 \to \mathcal{O} \to E \to F \to 0.$$

Assume first that F is locally free. Obtain the Riemann Roch formula by induction on the rank.

(iii) If F is not locally free, let T denote the torsion part of F, and let $\tilde{F} = F/T$. Consider the natural map

$$E \to F \to \tilde{F} \to 0$$

and let K denote its kernel. Show that M and $\tilde{F}$ are both locally free. Conclude the Riemann-Roch formula for E by induction on the rank.

2. (*Gonality.*) Let X be a smooth projective curve, and let $p \in X$. Show that there exists a surjective morphism $f : X \to \mathbb{P}^1$ of degree at most $g + 1$.

Hint: Construct f as a section of $\mathcal{O}_X((g + 1)p)$. To show that such an f exists, use Riemann-Roch.

Remark: The smallest degree of a morphism $f : X \to \mathbb{P}^1$ is called the gonality of the curve. Thus

$$\text{gon}\,(X) \leq g + 1.$$

Most curves of genus g have gonality roughly $\frac{g+3}{2}$, but other values are also possible:

– Gonality 1 means $X = \mathbb{P}^1$.
– Curves of gonality 2 admit a degree 2 morphism

$$f : X \to \mathbb{P}^1.$$

These are termed *hyperelliptic curves* (if $g \geq 2$).
– *Trigonal curves* admit a degree 3 morphism $f : X \to \mathbb{P}^1$.

3. (*Hyperelliptic curves.*)

(i) Let Z be a smooth projective *hyperelliptic curve* of genus $g \geq 2$ (i.e. a curve of gonality 2). Show that any morphism

$$f : Z \to \mathbb{P}^1$$

of degree 2 has $2g+2$ ramification points $a_1, \ldots, a_{2g+2}$.

(ii) Let $X \subset \mathbb{A}^2$ be the hyperelliptic curve

$$y^2 = (x - a_1) \cdot \ldots \cdot (x - a_{2g+2}).$$

Let Y be the curve

$$w^2 = (1 - za_1) \cdots (1 - za_{2g+2}).$$

Clearly

$$(x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^{g+1}} \right)$$

is an isomorphism between X and Y away from $x \neq 0$. Let Z denote the variety obtained by gluing X and Y along this isomorphism. Now, Z turns out to be a smooth projective curve. (Smoothness was checked in Math 203a, PSet 7; projectivity requires an argument, but this is not asked for here.)

Prove that there exists a degree 2 morphism $f : Z \to \mathbb{P}^1$ which is ramified at $2g+2$ points. Conclude that the hyperelliptic curve Z has genus g.

(iii) Show that $x^i \frac{dx}{y}$, $0 \leq i \leq g-1$ is a basis for $H^0(Z, K_Z)$.

Hint: You will have to show that $\omega_i = x^i \frac{dx}{y}$ is regular on X. The only issues are extending ω_i across the points a_j. To do so, you may wish to rewrite this form using the identity

$$y^2 = (x - a_1) \cdot \ldots \cdot (x - a_{2g+2}).$$

You will also need to check that ω_i is regular on Y.

4. (*Genus* 2 *curves.*) Let X be a smooth projective genus 2 curve.

(i) Show that X is hyperelliptic.

Hint: Using Riemann-Roch and Serre duality, show that K_X is globally generated. Show that $|K_X|$ induces a morphism $f : X \to \mathbb{P}^1$ of degree 2.

(ii) Show that X can be exhibited as a degree 5 curve in $\mathbb{P}^3$.

(iii) We have seen that genus 1 curves are cubics in $\mathbb{P}^2$. By contrast, show that a genus 2 curve can never be a complete intersection in any projective space.

Hint: Compute the canonical bundle of X, and show K_X is not very ample using (i).