Math 203, Problem Set 1. Due Friday, January 18.

1. (*Distinguished open sets.*) Let $X = \mathrm{Spec}(A)$ be an affine scheme. For each $f \in A$, recall the basic open sets

$$X_f = \{\mathfrak{p} \in X : f \notin \mathfrak{p}\}.$$

Show that

(i) $X_f \subset X_g$ iff $\sqrt{(f)} \subset \sqrt{(g)}$. In particular,

$$X_f = \emptyset \iff f \text{ is nilpotent},$$

and

$$X_f = X \iff f \text{ is a unit}.$$

(ii) Show that X is quasi-compact i.e. any cover by open subsets admits a finite subcover. More generally, each X_f is quasi-compact.

Hint: You may want to recall that the radical of an ideal $\mathfrak{a} \subset A$ is the intersection of prime ideals containing $\mathfrak{a}$.

2. (*Specialization and generization. Generic points.*) Given two points x, y of a topological space X, we say that x is a specialization of y, and y is a generization of x, if $x \in \overline{\{y\}}$.

A related notion is that of generic points: if Y is a closed subset of X, we say η_Y is a generic point of Y if $\overline{\{\eta_Y\}} = Y$.

(i) If $X = \mathrm{Spec}\ (A)$, show that $\mathfrak{q}$ is a specialization of $\mathfrak{p}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Hence show that

$$\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p}) = \{\mathfrak{q} : \mathfrak{p} \subset \mathfrak{q}\}.$$

Show that the closed points $\mathfrak{p}$ of X correspond to the maximal ideals of A.

(ii) Assume that $X = \mathrm{Spec}(A)$ is an affine scheme. Show that any nonempty irreducible closed subset $Y \subset X$ admits a unique generic point η_Y.

3. (*Stalks over generic points.*) Let X be an affine scheme, and let Y be an irreducible closed subset of X. If η_Y is the generic point of Y, we write $\mathcal{O}_{X,Y}$ for the stalk $\mathcal{O}_{X,\eta_Y}$.

Show that $\mathcal{O}_{X,Y}$ is "the ring of rational functions on X that are regular at a general point of Y", i.e. it is isomorphic to the ring of equivalence classes of pairs (U, ϕ), where $U \subset X$ is open with $U \cap Y \neq \emptyset$ and $\phi \in \mathcal{O}_X(U)$. Two such pairs (U, ϕ) and (U', ϕ') are called equivalent if there is an open subset $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ such that $\phi|_V = \phi'_V$.

In particular, if X is a scheme that is a variety, then $\mathcal{O}_{X,\eta_X}$ is the function field of X.

4. (*Morphisms of affine schemes.*) Let $f : A \to B$ be a morphism of rings, and let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $\mathfrak{q}$ is a point of Y, then $f^{-1}(\mathfrak{q})$ is a prime ideal in A hence a point of X. Therefore, we have a well-defined morphism

$$f^\star : Y \to X.$$

(i) Show that $f^\star$ is continuous.
(ii) If f is surjective with kernel $\mathrm{Ker}(f) \subset A$, show that $f^\star$ is homeomorphism of Y onto the closed subset $Z(\mathrm{Ker}(f))$ of X. In particular, show that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{n})$ are homeomorphic, where $\mathfrak{n}$ is the nilradical of A.
(iii) If f is injective, show that $f^\star$ is dominant. More precisely, the image $f^\star(Y)$ is dense in X iff $\mathrm{Ker}(f) \subset \mathfrak{n}$.

5. A field extension $K \subset L$ induces a morphism

$$\mathbb{A}^n_L \to \mathbb{A}^n_K.$$

We consider the case $K = \mathbb{Q}$ and $L = \overline{\mathbb{Q}}$. Consider the following points in $\mathbb{A}^2_{\overline{\mathbb{Q}}}$:

(i) $(\sqrt{2}, \sqrt{2})$;
(ii) $(\sqrt{2}, \sqrt{3})$;
(iii) $(x\sqrt{2} - y\sqrt{3})$.

What are the images of these points in $\mathbb{A}^2_{\mathbb{Q}}$ under the map $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$?

6. This requires a bit of number theory, in particular you may wish to recall Kummer factorization theorem. Consider the natural morphism

$$\mathrm{Spec}(\mathbb{Z}[\sqrt{3}]) \to \mathrm{Spec}(\mathbb{Z}).$$

What are the fibers over a prime (p) in $\mathrm{Spec}(\mathbb{Z})$? You may wish to discuss the following cases:

(i) $p = 2$ or $p = 3$;
(ii) the case when 3 is a quadratic residue mod p;
(iii) the case when 3 is not a quadratic residue mod p.

Use this to "draw" a picture of $\mathrm{Spec}(\mathbb{Z}[\sqrt{3}])$.

7. Let $f(x, y) = y^2 - x^2 - x^3$. Describe the affine scheme $X = \mathrm{Spec}\ A/(f)$ set-theoretically for the following rings A:

(i) $A = \mathbb{C}[x, y]$ (the standard polynomial ring),
(ii) $A = \mathbb{C}[x, y]_{(x,y)}$ (the localization of the polynomial ring at the origin),
(iii) $A = \mathbb{C}[[x, y]]$ (the ring of formal power series).

Interpret the results geometrically. In which of the three cases is X irreducible?

8. For each of these cases below give an example of an affine scheme X with that property, or prove that such an X does not exist:

(i) X has infinitely many points, and $\dim X = 0$;
(ii) X has exactly one point, and $\dim X = 1$;
(iii) X has exactly two points, and $\dim X = 1$;
(iv) $X = \operatorname{Spec} A$ with $A \subset \mathbb{C}[x]$, and $\dim X = 2$.