Math 203, Problem Set 2. Due Monday, January 28.

1. (*Sheaves of modules.*) Let X be a scheme. A sheaf $\mathcal{F} \to X$ is said to be a sheaf of $\mathcal{O}_X$-modules provided that for all open sets $U \subset X$, $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ in a fashion compatible with restrictions.

(i) Make this definition precise.

(ii) Now assume $X = \text{Spec } A$, and let M be an A-module. Show that the assignment

$$X_f \to \mathcal{F}(X_f) = M_f$$

defines a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.

(iii) Show furthermore that the stalks of this sheaf are

$$\mathcal{F}_{\mathfrak{p}} = M_{\mathfrak{p}}.$$

We will write $\widetilde{M}$ for this sheaf.

Remark: This is a *mandatory* problem, and we will use it later in the course. The proof is very similar to the construction of the structure sheaf given in class. Give as many details as you deem necessary.

2. Consider the arithmetic surface $X = \text{Spec } \mathbb{Z}[x] = \mathbb{A}^1_{\mathbb{Z}}$. Let $\mathfrak{p} = (2, x)$ be a point in X. Show that regular functions over $X \setminus \{\mathfrak{p}\}$ extend to X.

Remark: This should be paralleled with Hartogs' theorem for $\mathbb{A}^2_{\mathbb{C}}$ proved last quarter. Just as in the proof given last quarter, pick a regular function ϕ, cover X by two suitable distinguished open sets X_f and X_g and ask for agreement over overlaps.

3. Let $X = \mathbb{P}^1_{\mathbb{Z}}$. Show that the set of morphisms $\text{Spec } \mathbb{Z} \to X$ is $\mathbb{Q} \cup \{\infty\}$.

Hint: Cover X by the two standard affine sets $\text{Spec } \mathbb{Z}[x]$ and $\text{Spec } \mathbb{Z}[1/x]$ and consider their pre-images U and V in $\text{Spec } \mathbb{Z}$. The sets U, V are open in $\text{Spec } \mathbb{Z}$. What are the open sets in $\text{Spec } \mathbb{Z}$? Then impose agreement over overlaps.

4. Let X be a scheme. For each $x \in X$, write $k(x)$ for the residue field.

(i) Let K be a field. Show that to give a morphism $\text{Spec}(K) \to X$ is the same as giving a point of X and an inclusion $k(x) \to K$.

(ii) Show that there is a natural morphism $\text{Spec } \mathcal{O}_{X,x} \to X$. Show that the image can be identified with the intersection of all open subsets containing x.

5. (*Density of closed points.*) A scheme X is locally of finite type over a field k if it can be covered by affine open sets $U_i = \text{Spec } A_i$ with A_i a finitely generated k-algebra. Let X be a scheme locally of finite type over k.

(i) Show that x is a closed point iff the extension $k \subset k(x)$ is finite.

Hint: In one direction, recall the following version of the Nullstellensatz: for a maximal ideal $\mathfrak{m}$ in a finitely generated k-algebra A, $A/\mathfrak{m}$ is finite over k.

(ii) In particular, show that if k is algebraically closed, then the set of closed points of X can be identified with

$$X(k) = \mathrm{Hom}(\mathrm{Spec}\ k, X).$$

(iii) Show that if $x \in U \cap V$ for two affine open sets $U, V \subset X$, then x is closed in U iff x is closed in V.

(iv) Show that the set of closed points is dense in X (in fact, in every closed subset of X).

(v) On the other hand, give an example of a scheme whose closed points are not dense. *Hint:* DVR's.

6. (*The Zariski tangent space.*) Let X be a prevariety over a field k, and let $p \in X$ be a closed point of X. Let

$$D = \mathrm{Spec}\ k[\epsilon]/(\epsilon^2)$$

be the *double point*. Show that the tangent space $T_{X,p}$ to X at p can be identified with the set of morphisms $D \to X$ that map the unique point of D to p.

7. (*Reduced schemes.*) Show that for a scheme X the following are equivalent:

(i) X is *reduced* i.e. for all $U \subset X$ open, $\mathcal{O}_X(U)$ has no nilpotents;

(ii) for all $p \in X$, the local rings $\mathcal{O}_{X,p}$ have no nilpotents.