Math 203, Problem Set 4. Due Friday, February 22.

1. (*Quasi-coherent sheaves.*) Reading assignment (do not hand in): show that quasi-coherence is affine-local on the target. This is Theorem 13.2.1 in Vakil's notes or Proposition 5.4 in Hartshorne.

2. (*Coherent sheaves.*) Let X be a Noetherian scheme. Show that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free sheaves of finite rank on U.

3. (*Vector bundles over affine schemes.*)

(i) Show that every locally free sheaf of finite rank over $\mathbb{A}^1_k$ is free.

(ii) True or false: any locally free sheaf on an affine scheme is free.

4. (*Projection formula.*) Let $f : X \to Y$ be a morphism of schemes and $\mathcal{E}$ a locally free sheaf of finite rank over Y, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Show that

$$f_\star(f^\star\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_\star\mathcal{F}.$$

5. (*Pushforwards of coherent sheaves.*) Let $f : X \to Y$ be a morphism of Noetherian schemes and $\mathcal{F}$ a coherent sheaf on X.

(i) Show that the pushforward $f_\star\mathcal{F}$ may not be coherent.

Hint: You can take $X = \mathbb{A}^2$, $Y = \mathbb{A}^1$, for suitable $f, \mathcal{F}$.

(ii) Show that if f is a finite morphism, then $f_\star\mathcal{F}$ is coherent. In particular, this applies to closed immersions $\iota : X \to Y$, so that if $\mathcal{F}$ is coherent on X, then $\iota_\star\mathcal{F}$ is coherent on Y.

6. (*Criterion for local freeness.*) Let $\mathcal{F}$ be a coherent sheaf over a Noetherian scheme X. Define

$$\phi(x) = \dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$$

where $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_{X,x}$.

(i) Show that ϕ is upper-semicontinuous, that is the set $\{x \in X : \phi(x) \geq n\}$ is closed.

(ii) (Optional) If X is reduced, then $\mathcal{F}$ is locally free of rank r over X if and only if ϕ is constant equal to r.

Hint: You may assume $X = \mathrm{Spec} A$ is affine and $\mathcal{F} = \widetilde{M}$. For (i), pick generators m_i for M as an A-module. If $\phi(x) = n$, pick a basis $f_1, \ldots, f_n$ of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ and use Nakayama to lift to generators $f_1, \ldots, f_n$ of $\mathcal{F}_x$. Compare the f's and the m's in $\mathcal{F}_x$, and show that $f_1, \ldots, f_n$ generate $\mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y$ for y near x.