Math 203, Problem Set 6. Due Friday, March 15.

For this problem set, you may assume that all line bundles \( \mathcal{O}_{\mathbb{P}^r}(\ell) \) over projective space have no intermediate cohomology

\[
H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) = 0 \text{ for all } 0 < i < r \text{ and all } \ell.
\]

We will prove this in the last lecture.

1. (Affine morphisms.) Let \( f : X \to Y \) be an affine morphism between two Noetherian separated schemes, and let \( \mathcal{F} \to X \) be a quasicoherent sheaf. Show that

\[
H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F}).
\]

Hint: Consider the affine open cover \( \mathcal{U} \) of \( Y \), and the associated cover \( f^{-1}(\mathcal{U}) \) for \( X \).

2. (Hilbert polynomials and arithmetic genus.) Let \( \mathcal{F} \to X \) be a coherent sheaf over a projective variety \( X \subset \mathbb{P}^r_k \).

(i) Show that there exists a polynomial \( \chi_{\mathcal{F}} \) such that

\[
\chi(\mathcal{F}(n)) = \chi_{\mathcal{F}}(n).
\]

Hint: First reduce to the case \( X = \mathbb{P}^r \) by considering \( \iota_*\mathcal{F} \). Over \( \mathbb{P}^r \), you may wish to argue by induction on \( r \), using the exact sequence

\[
0 \to \mathcal{F}(-H) \to \mathcal{F} \to \mathcal{F}|_H \to 0
\]

for a suitable hyperplane \( H \subset \mathbb{P}^r \). The only issue is exactness on the left. You can construct a suitable hyperplane working over affine patches and using a bit of commutative algebra (associated points).

(ii) In particular, if \( \mathcal{F} = \mathcal{O}_X \) this recovers the Hilbert polynomial we introduced in Math 203a.

(iii) Using (ii), show that the arithmetic genus of \( X \) is defined as

\[
p_a(X) = (-1)^{\dim X - 1}(\chi(\mathcal{O}_X) - 1).
\]

In particular, if \( X \) is an irreducible projective curve then

\[
p_a(X) = \dim H^1(X, \mathcal{O}_X).
\]

This definition is independent of the projective embedding.

3. (Hypersurfaces.) Let \( X \subset \mathbb{P}^r \) be a smooth hypersurface of degree \( d \) in projective space. Show that

(i) for all \( \ell \), the map

\[
H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) \to H^0(X, \mathcal{O}_X(\ell))
\]

is surjective. This was stated as a fact in a previous problem set, but now you have the tools to construct a proof.
**Hint:** You may wish to begin with the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^r} \rightarrow \mathcal{O}_X \rightarrow 0.$$  

(ii) In particular, setting $\ell = 0$, $X$ is connected.

(iii) Show that the intermediate cohomology of the twisting line bundles vanishes

$$H^i(X, \mathcal{O}_X(\ell)) = 0, \quad 0 < i < \dim X, \quad \text{for all } \ell.$$  

**Remark:** A sheaf $\mathcal{F}$ is said to be ACM (arithmetically Cohen-Macaulay) if $\mathcal{F}(\ell)$ has no intermediate cohomology for all $\ell$. Over complete intersections, $\mathcal{O}_X$ is ACM.

(iv) Using problem 2, show that the arithmetic genus $p_a(X) = \dim H^d(X, \mathcal{O}_X)$.

**Remark:** By induction, similar statements holds true for complete intersections in projective space. The proofs are identical.

4. **(Hodge numbers.)** For a smooth projective variety over $k$, set $H^{p,q}(X) = H^q(X, \Omega^p_X)$ and define the Hodge numbers

$$h^{p,q}(X) = \dim_k H^{p,q}(X).$$

Show that for projective space $h^{p,q}(\mathbb{P}^n) = 0$ if $p \neq q$ and $h^{p,q}(\mathbb{P}^n) = 1$ if $p = q$.

**Remark:** If $X$ is a smooth complex projective variety, then the cohomology of $X$ splits as

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

This is called the Hodge decomposition. The Hodge numbers $h^{p,q}$ satisfy the symmetries

$$h^{p,q} = h^{q,p} \quad \text{and} \quad h^{p,q} = h^{d-p,d-q}.$$  

The latter symmetry follows from Serre duality. The Betti numbers of $X$ can be calculated from the Hodge numbers

$$b_k(X) = \sum_{p+q=k} h^{p,q}.$$  

The Hodge numbers (arranged in a rotated square) form the **Hodge diamond** of $X$.

**Hint:** You may wish to start with the Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \mathbb{C}^{n+1} \rightarrow \mathcal{O} \rightarrow 0.$$  

It is important to show that if

$$0 \rightarrow E \rightarrow F \rightarrow \mathcal{O} \rightarrow 0$$

is an exact sequence of vector bundles then you also have exactness of

$$0 \rightarrow \Lambda^p E \rightarrow \Lambda^p F \rightarrow \Lambda^{p-1} E \rightarrow 0.$$
To construct the last map in the exact sequence above, let \( \pi : F \to \mathcal{O} \), and note that for all \( f_1, f_2 \) local sections of \( F \) we have \( \pi(f_1)f_2 - \pi(f_2)f_1 \) is a local section of \( E \). Then define

\[
\text{Prove the surjectivity of the map and check exactness by comparing ranks.}
\]

5. (The Picard group.) Let \( X \) be a variety. Show that the Pic\((X)\) can be identified with \( H^1(X, \mathcal{O}_X^*) \) where \( \mathcal{O}_X^* \) is the sheaf of nowhere-zero regular functions.

\textit{Hint:} Start with trivializations \( \psi_i \) of a line bundle \( L_{|U_i} \to \mathcal{O}_{U_i} \) over open sets \( U_i \), and consider \( \psi_{ij} = \psi_i \circ \psi_j^{-1} \) over \( U_{ij} \). Show that \( \psi_{ij} \) satisfy the cocycle condition and thus define an element in \( H^1(X, \mathcal{O}_X^*) \).

6. (On finiteness of cohomology.) Let \( X = \mathbb{A}^2_k \setminus \{(0,0)\} \). Show that \( \dim_k H^1(X, \mathcal{O}_X) \) is infinite.

\textit{Hint:} Use the Cech cover with two open sets \( U = \{x \neq 0\} \) and \( V = \{y \neq 0\} \).