Math 203, Problem Set 6. Due Friday, March 15.

For this problem set, you may assume that *all* line bundles $\mathcal{O}_{\mathbb{P}^r}(\ell)$ over projective space have no intermediate cohomology

$$H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell) = 0 \text{ for all } 0 < i < r \text{ and all } \ell.$$

We will prove this in the last lecture.

1. (*Affine morphisms.*) Let $f : X \to Y$ be an affine morphism between two Noetherian separated schemes, and let $\mathcal{F} \to X$ be a quasicoherent sheaf. Show that

$$H^i(X, \mathcal{F}) = H^i(Y, f_\star \mathcal{F}).$$

Hint: Consider the affine open cover $\mathfrak{U}$ of Y, and the associated cover $f^{-1}(\mathfrak{U})$ for X.

2. (*Hilbert polynomials and arithmetic genus.*) Let $\mathcal{F} \to X$ be a coherent sheaf over a projective variety $X \subset \mathbb{P}^r_k$.

(i) Show that there exists a polynomial $\chi_{\mathcal{F}}$ such that

$$\chi(\mathcal{F}(n)) = \chi_{\mathcal{F}}(n).$$

Hint: First reduce to the case $X = \mathbb{P}^r$ by considering $\iota_\star \mathcal{F}$. Over $\mathbb{P}^r$, you may wish to argue by induction on r, using the exact sequence

$$0 \to \mathcal{F}(-H) \to \mathcal{F} \to \mathcal{F}|_H \to 0$$

for a suitable hyperplane $H \subset \mathbb{P}^r$. The only issue is exactness on the left. You can construct a suitable hyperplane working over affine patches and using a bit of commutative algebra (associated points).

(ii) In particular, if $\mathcal{F} = \mathcal{O}_X$ this recovers the Hilbert polynomial we introduced in Math 203a.

(iii) Using (ii), show that the arithmetic genus of X is defined as

$$p_a(X) = (-1)^{\dim X - 1}(\chi(\mathcal{O}_X) - 1).$$

In particular, if X is an irreducible projective curve then

$$p_a(X) = \dim H^1(X, \mathcal{O}_X).$$

This definition is independent of the projective embedding.

3. (*Hypersurfaces.*) Let $X \subset \mathbb{P}^r$ be a smooth hypersurface of degree d in projective space. Show that

(i) for all ℓ, the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) \to H^0(X, \mathcal{O}_X(\ell))$$

is surjective. This was stated as a fact in a previous problem set, but now you have the tools to construct a proof.

Hint: You may wish to begin with the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r}(-d) \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X \to 0.$$

(ii) In particular, setting $\ell = 0$, X is connected.

(iii) Show that the intermediate cohomology of the twisting line bundles vanishes

$$H^i(X, \mathcal{O}_X(\ell)) = 0, \;\; 0 < i < \text{dim X}, \;\; \text{for all } \ell.$$

Remark: A sheaf $\mathcal{F}$ is said to be ACM (arithmetically Cohen-Macaulay) if $\mathcal{F}(\ell)$ has no intermediate cohomology for all ℓ. Over complete intersections, $\mathcal{O}_X$ is ACM.

(iv) Using problem 2, show that the arithmetic genus $p_a(X) = \dim H^d(X, \mathcal{O}_X)$.

Remark: By induction, similar statements holds true for complete intersections in projective space. The proofs are identical.

4. (*Hodge numbers.*) For a smooth projective variety over k, set $H^{p,q}(X) = H^q(X, \Omega^p_X)$ and define the Hodge numbers

$$h^{p,q}(X) = \dim_k H^{p,q}(X).$$

Show that for projective space $h^{p,q}(\mathbb{P}^n) = 0$ if $p \neq q$ and $h^{p,q}(\mathbb{P}^n) = 1$ if $p = q$.

Remark: If X is a smooth complex projective variety, then the cohomology of X splits as

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

This is called the Hodge decomposition. The Hodge numbers $h^{p,q}$ satisfy the symmetries

$$h^{p,q} = h^{q,p} \text{ and } h^{p,q} = h^{d-p,d-q}.$$

The latter symmetry follows from Serre duality. The Betti numbers of X can be calculated from the Hodge numbers

$$b_k(X) = \sum_{p+q=k} h^{p,q}.$$

The Hodge numbers (arranged in a rotated square) form the *Hodge diamond* of X.

Hint: You may wish to start with the Euler sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \mathbb{C}^{n+1} \to \mathcal{O} \to 0.$$

It is important to show that if

$$0 \to E \to F \to \mathcal{O} \to 0$$

is an exact sequence of vector bundles then you also have exactness of

$$0 \to \Lambda^p E \to \Lambda^p F \to \Lambda^{p-1} E \to 0.$$

To construct the last map in the exact sequence above, let $\pi : F \to \mathcal{O}$, and note that for all f_1, f_2 local sections of F we have $\pi(f_1)f_2 - \pi(f_2)f_1$ is a local section of E. Then define

$$f_1 \wedge \ldots \wedge f_p \mapsto \wedge_{j=2}^{p}(\pi(f_1)f_j - \pi(f_j)f_1).$$

Prove the surjectivity of the map and check exactness by comparing ranks.

5. (*The Picard group.*) Let X be a variety. Show that the $\mathrm{Pic}(X)$ can be identified with $H^1(X, \mathcal{O}_X^\star)$ where $\mathcal{O}_X^\star$ is the sheaf of nowhere-zero regular functions.

Hint: Start with trivializations ψ_i of a line bundle $L|_{U_i} \to \mathcal{O}_{U_i}$ over open sets U_i, and consider $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ over U_{ij}. Show that ψ_{ij} satisfy the cocycle condition and thus define an element in $H^1(X, \mathcal{O}_X^\star)$.

6. (*On finiteness of cohomology.*) Let $X = \mathbb{A}_k^2 \setminus \{(0,0)\}$. Show that $\dim_k H^1(X, \mathcal{O}_X)$ is infinite.

Hint: Use the Cech cover with two open sets $U = \{x \neq 0\}$ and $V = \{y \neq 0\}$.