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- Other useful texts are:
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Figure: Grothendieck
Course outline

- Prime spectra of rings
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▶ Prime spectra of rings

▶ Schemes and their properties
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- Morphisms and their properties
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- **Prime spectra** of rings
- **Schemes** and their properties
- **Morphisms** and their properties
- Line bundles and **divisors**
- Coherent sheaves
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- Examples and computations
  - projective space
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*Scheme theoretic intersection*
Issues to address

- Reducible objects

Nilpotents
Radical ideals correspond to affine algebraic sets $X \subset \mathbb{A}^n$

What do non-radical ideals correspond to?

Arbitrary rings $A(X)$ is f.g. $k$-algebra

What do algebras which are not finitely generated give?

Over non-algebraically closed fields?

What if there's no field at all?
Issues to address

- **Reducible** objects – we dealt with them component by component.
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- Outcome: **Schemes**
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▶ Outcome: Schemes
General principles

(A) Think relatively
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- Families of objects

\[ f : X \to S. \]
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General principles

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- For varieties \( X/k \) – structure morphism
  \[ f : X \to \text{“}k\text{”} \]
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(B) The base may change
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- Basechange (draw diagram)
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General principles

(C) Think categorically

We think of an object $X$ in relation to all other schemes $S$.

Functor of points $h_X: \text{Schemes} \to \text{Sets}$, $h_X(S) = \text{Mor}(S, X)$

To know $X$ is tantamount to knowing $h_X$.

We used this to define $X \times Y$.

We will generalize to fibered products.

$S$-points $\iff$ morphisms $S \to X$

for varieties: "old" points $\iff$ $k$-points
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The very idea of scheme is of infantile *simplicity* – so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really “not serious”.

A. Grothendieck
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C. McLarty
Rough Strategy

- **Affine** schemes – commutative rings
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Roadmap for affine schemes
Rough Strategy

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Roadmap for affine schemes

A commutative ring with unit
Rough Strategy

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A commutative ring with unit

- as a set $X = \text{Spec}A$
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A commutative ring with unit

- as a **set** $X = \text{Spec}A$
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Roadmap for affine schemes

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- as a **set** $X = \text{Spec}A$
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- sheaf of regular functions - $\mathcal{O}_X$
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**Roadmap for affine schemes**

* A commutative ring with unit
  - as a set \( X = \text{Spec} A \)
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**Example:** \( A = k[x_1, \ldots, x_n]/I(X) \) should “recover” varieties
Rough Strategy

- **Affine schemes** – commutative rings
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- **Define** morphisms

Roadmap for affine schemes

A commutative ring with unit

- as a **set** \( X = \text{Spec} A \)
- as a **topological space** - Zariski topology
- **sheaf** of regular functions - \( \mathcal{O}_X \)

Example: \( A = k[x_1, \ldots, x_n]/I(X) \) should “recover” varieties

Philosophy: Elements of \( A \) are regular functions on \( X \).
Rough Strategy

- **Affine** schemes – commutative rings
- **Glue** affine schemes – arbitrary schemes
- Define morphisms

Roadmap for affine schemes

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Let $A$ be a ring. $X = \text{Spec } A$ consists in all prime ideals in $A$. 

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$p$ prime $\implies$ $A/p$ domain $\implies$ $k(p)$ fraction field
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Values in different fields!!!

Example:

- $A = k[x_1, \ldots, x_n], f \in A$

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Varieties vs. Schemes - Part I

- \( X \subset \mathbb{A}^n \) variety
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If $x \in X$ then $m_x \subset A(X)$ maximal ideal gives a point of $X^\text{sch}$

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Zariski topology

In 203A, for a set $S$ of polynomials, we defined the closed subsets

$$Z(S) = \{ x : f(x) = 0 \text{ for all } f \in S \}.$$
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\( f_1 \in S_1 \setminus p, f_2 \in S_2 \setminus p \)

\( f = f_1 f_2 \in S_1 \cap S_2 \)

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Distinguished open sets

Let $f \in A$, $X = \text{Spec } A$.

- The open sets

$$X_f = X \setminus Z(f) = \{p \in X : f \notin p\}$$

are called distinguished.
Distinguished open sets

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