Math 206 - Lecture 1

January 6, 2020

0 Logistics

- Zoom lectures WF 1 - 2:15 PM.
- Attendance vs. homework / final project
- Prerequisites
  - A.G. at the level of Math 203
  - cohomology of sheaves
  - ample, very ample, basepointfree ...
  - Chow, Chern classes
  - Grothendieck - Riemann - Roch ...

References

Huybrechts

Lectures on K3 surfaces

Barth – Hulek – Peters – van de Ven

Compact Complex Surfaces, VII

Beauville ...

Géométrie des surfaces K3 : modules et périods

Course outline .

[I] examples of K3s

[II] linear series on K3s

[III] elliptic K3s

[IV] moduli of K3s.

[V] tautological classes over the moduli space

[1] What are K3 surfaces

Definition X smooth projective surface $/\mathbb{C}$ such that $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

$$q = \dim H^1(X, \mathcal{O}_X) = \text{irregularity.}$$

Examples [i]. $X \hookrightarrow \mathbb{P}^3$, X = quartic surface

[ii] $X \longrightarrow \mathbb{P}^2$ double cover branched sextic

[iii] $X \hookrightarrow \mathbb{P}^5$, $X = Q_1 \cap Q_2 \cap Q_3$

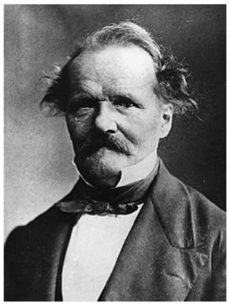
We will return to these examples in the next lecture.

Remark (analytic K3 surfaces)

One can study X = complex manifold of dim 2.

We will not consider these. Siu (1983) : Kähler.

André Weil (1958): ... *il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire...*



Kummer

Kähler

Kodaira

How general are the surfaces we wish to study?

Classification of curves of genus g.

$$H^0(C,\mathbb{Z}) = \mathbb{Z}, \quad H^1(C,\mathbb{Z}) = \mathbb{Z}^{2g}, \quad H^2(C,\mathbb{Z}) = \mathbb{Z}.$$

$g = 0$ $\qquad C \cong \mathbb{P}^1$

$g = 1$ $\qquad K_C \cong \mathcal{O}_C$

$g \geq 2$ $\qquad$ most curves

Kodaira dimension Consider the smallest κ such that

$$\frac{h^0(X, K_X^{\otimes m})}{m^\kappa} \text{ bounded.}$$

$\kappa = -\infty$ $\qquad C \cong \mathbb{P}^1$

$\kappa = 0$ $\qquad C$ genus 1

$\kappa = 1$ $\qquad C$ genus $g \geq 2$.

Riemann – Roch

Indeed, $h^0(X, K_X^{\otimes m}) = 1-g + m(2g-2) \sim m^{\kappa=1}$.

Surfaces — Enriques – Kodaira classification

X minimal. Coarse classification:

$\kappa = -\infty$: $q = 0$: $\mathbb{P}^2$, $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$, $n \neq 1$

$q \neq 0$: $X \longrightarrow C$ ruled surface

$\kappa = 0$: $\swarrow$ K3 surfaces + others

$\kappa = 1$: $X \xrightarrow{\pi} C$, general fiber is elliptic curve

$\kappa = 2$: X surfaces of general type

When $\kappa = 0$, finer classification:

- $K_X \cong \mathcal{O}_X$ $\boxed{\text{I}}$ $q = 0$: X = K3 surface

$\boxed{\text{II}}$ $q \neq 0$: X = abelian

• $K_X \not\cong \mathcal{O}_X$

$\boxed{i}$ $q = 0 \Rightarrow X =$ Enriques surface, $X = K3/\mathbb{Z}_2$, $K_X^{\otimes 2} \cong \mathcal{O}_X$

$\boxed{ii}$ $q \neq 0 \Rightarrow X =$ bielliptic surface, $K_X^{\otimes m} \cong \mathcal{O}_X$

$m = 2, 3, 4, 6$

$X = E \times F / G$, E, F elliptic curves

G finite group, $G \subseteq E$ acts by translations

Why study K3 surfaces?

$\boxed{i}$ interesting for both classical & not-so-classical algebraic geometry

$\boxed{ii}$ arithmetic, differential geometry, topology, dynamics

It is hard to match the geometric beauty of K3 surfaces

[2] What will our attitude be?

Pursue analogies with curves. Such analogies will be evident in the choice of results we will cover:

[i] linear series

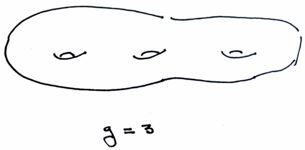
[ii] Torelli thm

[iii] moduli theory

Remark A different possible comparison is between K3 surfaces & abelian surfaces.

Curves

- Let C be a smooth compact complex curve of genus g



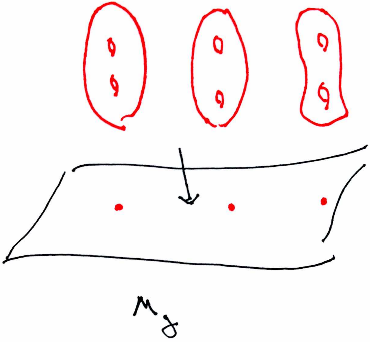
- The topology of C is fixed by the genus

$$H^0(C, \mathbb{Z}) = \mathbb{Z}, \;\; H^1(C, \mathbb{Z}) = \mathbb{Z}^{2g}, \;\; H^2(C, \mathbb{Z}) = \mathbb{Z}$$

- The complex structure of X is allowed to vary

Moduli of curves

- $\mathcal{M}_g$ is the moduli space of all smooth genus $g \geq 2$ curves



Selected facts about the moduli of curves

- $\mathcal{M}_g$ is irreducible of dimension $3g - 3$
- smooth complex orbifold
- Pic $(\mathcal{M}_g)$ has rank 1

Cohomology

- Harer: cohomology stabilizes

$$H^k(\mathcal{M}_\infty) = \lim_{g \to \infty} H^k(\mathcal{M}_g)$$

- Mumford-Madsen-Weiss

$$H^\star(\mathcal{M}_\infty, \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots]$$

- The universal curve $\pi : \mathcal{C} \to \mathcal{M}_g$ has $\pi^{-1}([C]) \simeq C$. Set

$$\kappa_i = \pi_\star \left(c_1(\Omega_\pi)^{i+1} \right)$$

Cohomology

- For finite g, define the tautological cohomology

$$R^\star(\mathcal{M}_g) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots]/\text{relations}$$

- There is odd cohomology and there are non-tautological classes

Structure of tautological rings

- Question: How do we get relations between the κ's?
- Question: Can we write them in closed form?
- Question: Study the structure of the tautological rings?

Poincare Duality (PD)?

- Faber's conjectures

$$\mathrm{R}^k(\mathcal{M}_g) = 0 \text{ for } k > g-2, \quad \mathrm{R}^{g-2}(\mathcal{M}_g) = \mathbb{Q}$$

- perfect pairing

$$\mathrm{R}^k(\mathcal{M}_g) \times \mathrm{R}^{g-2-k}(\mathcal{M}_g) \to \mathbb{Q}$$

- evaluation of top monomials in κ's

$$\kappa_{a_1}\kappa_{a_2}\cdots\kappa_{a_m} \quad \text{for } \sum a_i = g-2$$

Poincare Duality (PD)?

- however $\mathcal{M}_g$ is not compact and not of dimension $g-2$
- any complete subvariety has dimension $\leq g-2$

Faber-Zagier relations

- consider two sets of formal variables

$$p_3, p_6, p_9, \ldots \text{ and } p_1, p_4, p_7, \ldots$$

- consider two hypergeometric series

$$A(t) = \sum_{k=0}^{\infty} \frac{(6k)!}{(2k)!(3k)!} t^k, \quad B(t) = \sum_{k=0}^{\infty} \frac{(6k)!}{(2k)!(3k)!} \frac{6k+1}{6k-1} t^k$$

- $\Psi(t,p) = (1 + tp_3 + t^2 p_6 + \ldots)A(t) + (p_1 + tp_4 + t^2 p_7 + \ldots)B(t)$

Faber-Zagier relations

- Expand

$$\log \Psi = \sum_{i,\sigma} c_{i,\sigma} t^i p^\sigma$$

- for i in a suitable range, the coefficient of $t^i p^\sigma$ in the expression

$$\exp\left(\sum_{i,\sigma} c_{i,\sigma} \kappa_i t^i p^\sigma\right) = 0$$

- the proof requires modern techniques

K3 Surfaces

- A K3 surface X is a simply connected smooth projective surface with $K_X = \mathcal{O}_X$

The topology of K3s

- The differentiable manifold underlying all $K3$ surfaces is always the same
- Cohomology groups

$$H^0(X,\mathbb{Z}) = \mathbb{Z}, \quad H^2(X,\mathbb{Z}) = \mathbb{Z}^{22}, \quad H^4(X,\mathbb{Z}) = \mathbb{Z}$$

Moduli space of K3s

- $\mathcal{F}_{2\ell}$ is the moduli space of $K3$ surfaces (X, H) of degree 2ℓ.
  - X is a $K3$ surface
  - $H \to X$ is primitive ample line bundle, $H^2 = 2\ell$
- dimension 19, not compact

Cohomology

Goal: Study the cohomology of $\mathcal{F}_{2\ell}$

Question: Does the cohomology stabilize?

- No:

$$\lim_{\ell\to\infty} \dim H^2(\mathcal{F}_{2\ell}) = \infty$$

- related to vector-valued cusp forms for metaplectic group

More on cohomology

- There is odd cohomology, e.g. for $\ell = 1$

$$P^{\text{odd}}(\mathcal{F}_2) = t^{27} + t^{31} + t^{33} + 2t^{35} + 2t^{37}$$

Goal: Define tautological classes over $\mathcal{F}_{2\ell}$

Goal: Find the structure of the tautological cohomology

Question: Construct algebraic non-tautological classes

κ-classes

- Let $\pi : (\mathcal{X}, \mathcal{H}) \to \mathcal{F}_{2\ell}$ be the universal surface
- Define

$$\kappa_{m,n} = \pi_\star \left(c_1(\mathcal{H})^m \cdot c_2(T_\pi)^n \right)$$

- The κ-ring

$$\kappa^\star(\mathcal{F}_{2\ell}) = \mathbb{Q}[\kappa_{m,n}]/\text{relations}$$

Question: How do we find relations between the $\kappa_{m,n}$'s?

Socle Conjectures

- there are even larger tautological rings

$$\kappa^\star(\mathcal{F}_{2\ell}) \subset R^\star(\mathcal{F}_{2\ell}) \subset H^\star(\mathcal{F}_{2\ell})$$

- Peterson and van der Geer showed that

$$R^{18}(\mathcal{F}_{2\ell}) = R^{19}(\mathcal{F}_{2\ell}) = 0, \;\; R^{17}(\mathcal{F}_{2\ell}) \neq 0$$

- compact subvarieties have dimension ≤ 17

Socle Conjectures

- Question: Is the vanishing

$$R^{18}(\mathcal{F}_{2\ell}) = R^{19}(\mathcal{F}_{2\ell}) = 0$$

  true in Chow?

- Question: Is it true that

$$R^{17}(\mathcal{F}_{2\ell}) = \mathbb{Q}?$$

- Question: If so, evaluate top monomials in κ's

Poincare duality?

Take $\ell = 1$.

Kirwan-Lee computed

$$P^{\text{even}}(\mathcal{F}_2) = 1 + 2t^2 + 3t^4 + 5t^6 + 6t^8 + 8t^{10} + 10t^{12} + 12t^{14}$$

$$+13t^{16} + 14t^{18}$$

$$+12t^{20} + 10t^{22} + 8t^{24} + 6t^{26} + 5t^{28} + 3t^{30} + 2t^{32}$$

Poincare duality?

Take $\ell = 1$.

Correction

$$P^{\text{even}}(\mathcal{F}_2) = 1 + 2t^2 + 3t^4 + 5t^6 + 6t^8 + 8t^{10} + 10t^{12} + 12t^{14}$$

$$+13t^{16} + 14t^{18}$$

$$+12t^{20} + 10t^{22} + 8t^{24} + 6t^{26} + 5t^{28} + 3t^{30} + 2t^{32} + t^{34}$$

Questions: Structure of the tautological rings (for other classes of surfaces as well)?

- Are there methods of obtaining relations?
- Do we obtain all relations this way? Write the relations in closed form (hypergeometric series, modular forms, etc)?
- Carry out explicit calculations in the tautological ring

Math 206 - Lecture 2

January 8, 2020

- Notes are available in Canvas – "Files"

- Plan for the first few lectures:

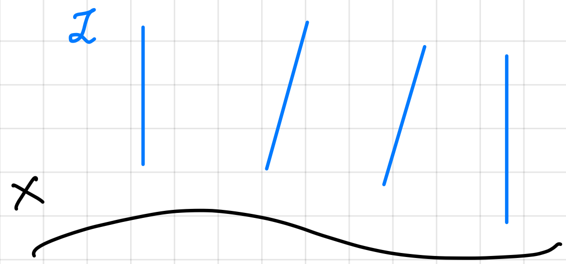
  (−1) Overview ⇝ last time

  (0) Review ⇝ today

  (1) Examples of K3 surfaces ⇝ next time

General facts that we will use often / $\mathbb{C}$

Curves Let X be a smooth projective curve



genus $X = g$

$\mathcal{L} \to X$ line bundle

$\boxed{a}$ Riemann–Roch

$$\chi(X, \mathcal{L}) = h^0(X, \mathcal{L}) - h^1(X, \mathcal{L})$$

$$= 1 - g + \deg \mathcal{L}$$

$\boxed{b}$ Serre duality $H^i(X, \mathcal{L})^\vee \cong H^{1-i}(X, \mathcal{L}^\vee \otimes K_X)$.

$\boxed{c}$ Kodaira vanishing : $\deg \mathcal{L} > 0$

$$H^1(X, K_X \otimes \mathcal{L}) = H^0(X, \mathcal{L}^\vee)^\vee = 0.$$

because $\mathcal{L}^\vee$ has negative degree so no sections

These theorems extend.

X smooth projective, $\dim X = d$.

a) Hirzebruch - Riemann - Roch

$$\chi(X, \mathcal{L}) = \sum_{k=0}^{d} (-1)^k h^k(X, \mathcal{L}) =$$

$$= \deg\left(e^{c_1(\mathcal{L})} \cdot \mathrm{todd}(X)\right).$$

requires more

b) Serre duality

$$H^i(X, \mathcal{L})^\vee = H^{d-i}(X, K_X \otimes \mathcal{L}^\vee)$$

c) Kodaira vanishing

$H^i(X, K_X \otimes \mathcal{L}) = 0$ if $\mathcal{L}$ ample, $i > 0$.

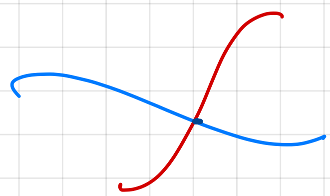
Surfaces

X smooth projective surface.

Intersection product C, D two divisors on X

$$C . D = \text{intersection product}$$

- symmetric $C . D = D . C$
- additive $(C_1 + C_2) . D = C_1 . D + C_2 . D$
- $C . D = \#(C \cap D)$ if C, D smooth intersecting transversally



- passes through rational equivalence.

$$C \equiv C', \ D \equiv D' \Rightarrow C . D = C' . D'$$

Remark Many possible definitions.

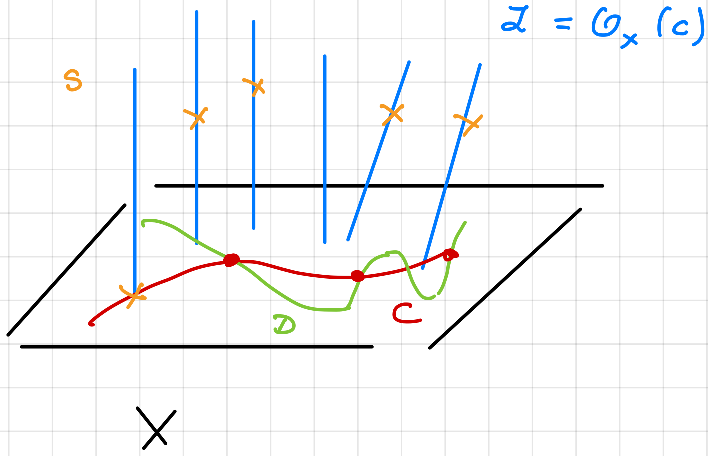
Hartshorne: D integral.

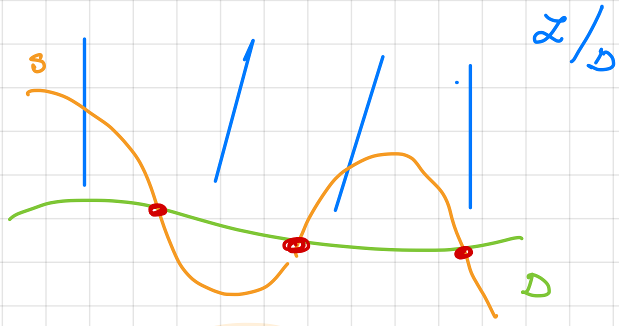
$$C . D = \deg [\underbrace{\mathcal{O}_X(C)/_D}_{\text{line bdle on } D}]$$

line bdle on D. $\longrightarrow$ divisor on D

$\rightsquigarrow$ points on X.

Why does this make sense?





$\deg \underbrace{\mathcal{O}_X(C)}_{S'}/_D.$

C effective, D smooth.

$\mathcal{O}_X(C) \rightsquigarrow$ section s cutting out C

$\mathcal{O}_X(C)/_D \rightsquigarrow$ section $s/_D$ cutting out $C \cap D$.

degree = # zeroes of sections. = # $C \cap D$

This works if C, D are smooth transverse.

Remark

In general, any $C \equiv C_1 - C_2$, C_1, C_2 smooth

H ample, $C_1 \in |nH|$, $n >> 0$. smooth by Bertini

$C_2 \in |C + nH|$, $n >> 0$. smooth & transverse to C_1

$\Downarrow$

$C \equiv C_2 - C_1$

Hirzebruch – Riemann – Roch

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{L \cdot (L - K_X)}{2}$$

Additional statement (Noether's formula)

$$\chi(X, \mathcal{O}_X) = \frac{K_X^2 + e_{top}(X)}{12}$$

Example $X = K3 \implies K_X \cong \mathcal{O}_X$

Serre duality

$$h^2(X, \mathcal{O}_X) = h^0(X, K_X) = h^0(X, \mathcal{O}_X) = 1$$

$$\implies \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X).$$

definition

$$= 1 - 0 + 1 = 2.$$

By Noether formula,

$$2 = \chi(X, \mathcal{O}_X) = \frac{\overset{=0}{K_X^2} + e_{top}(X)}{12} \implies e_{top}(X) = 24$$

$$H^1(X, \mathbb{C}) = 0 \Leftarrow H^1(X, \mathcal{O}_X) = 0, \quad H^3(X, \mathbb{C}) = 0.$$

$$H^0(X, \mathbb{C}) = H^4(X, \mathbb{C}) = \mathbb{C}$$

$$H^2(X, \mathbb{C}) = 22 \text{ diml.}$$

Important fact about exact sequences

$\det E = \Lambda^r E$ (rank 1), $E \to X$ vector bundle, $\operatorname{rk} E = r$.

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

(1) $\det E \cong \det E' \otimes \det E''$. Math 203b.

(2) $\chi(X, E) = \chi(X, E') + \chi(X, E'')$ Math 203b.

Adjunction (H. V).

$C \hookrightarrow X$ smooth curve, X smooth projective surface

Normal sequence

$$0 \longrightarrow T_C \longrightarrow T_X\big|_C \longrightarrow N_{C/X} \cong \mathcal{O}_X(C)\big|_C \longrightarrow 0.$$

$\Rightarrow$ by (1): $K_C = K_X\big|_C \otimes \mathcal{O}_X(C)\big|_C = (K_X + C)\big|_C$

Take degrees

$$2g-2 = (K_X + C).C$$

Remark This holds for all $Y \hookrightarrow X$ smooth, codim 1.

$$K_Y = (K_X + [Y])\big|_Y$$

Proof of Hirzebruch-Riemann-Roch X smooth proj

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + \frac{L.(L-K_X)}{2}$$

By previous remark, $\mathcal{L} = \mathcal{O}_X(C-D)$, C, D smooth

Two exact sequences

$$(1) \quad 0 \to \mathcal{L} = \mathcal{O}_X(C-D) \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)/_D \to 0.$$

$$(2) \quad 0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)/_C \to 0.$$

Take Euler characteristics

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X(C)) - \chi(X, \mathcal{O}_X(C)/_D) \quad \text{by } (1)$$

$$= \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C)/_C) - \chi(\mathcal{O}_X(C)/_D) \quad \text{by } (2)$$

Riemann-Roch for C, D

$$= \chi(\mathcal{O}_X) + (1 - g_C + C^2) - (1 - g_D + C.D)$$

adjunction

$$= \chi(\mathcal{O}_X) + \left(-\frac{C^2 + C.K_X}{2} + C^2\right) - \left(-\frac{D^2 + D.K_X}{2} + D^2\right)$$

$$= \chi(\mathcal{O}_X) + \frac{L.(L-K_X)}{2}$$

For K3 surfaces C smooth curve on $X = K3$

$\boxed{i}$ $C \hookrightarrow X$, $2\,\text{genus}(C) - 2 = C^2 + C.K_X = C^2$

$$\Rightarrow \text{genus}(C) = 1 + \frac{C^2}{2}$$

$\boxed{ii}$ $L \to X$, $\chi(X, L) = \chi(X, \mathcal{O}_X) + \dfrac{L(L-K_X)}{2} =$

$$= 2 + \frac{L^2}{2}.$$

Important construction

If L is very ample, $i : X \xrightarrow{|L|} \mathbb{P} H^0(X, L)$

$$h^0(X, L) = h^0(X, L) - \overbrace{h^1(X, L)}^{0} + \overbrace{h^2(X, L)}^{0}.$$

$$= \chi(X, L) = 2 + \frac{L^2}{2}.$$

Note that $h^1(X, L) = h^1(X, K_X \otimes L) = 0$ by Kodaira &

$h^2(X, L) = 0$

Write $L^2 = 2g - 2$. $\Rightarrow \chi(X, L) = g + 1$ $\therefore$ $i : X \hookrightarrow \mathbb{P}^g$

Let H general hyperplane in $\mathbb{P}^g$. Let $C = X \cap H$

$\Downarrow$

$\mathcal{O}_X(C) \cong \mathcal{L}$

Claim C has genus g.

$$\text{genus}(C) = 1 + \frac{C^2}{2} = 1 + \frac{\mathcal{L}^2}{2} = 1 + (g-1) = g.$$

Remark Restricting to C: $i|_C : C \longrightarrow \mathbb{P}^g \cap H \cong \mathbb{P}^{g-1}$.

$$i|_C^* \mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{L}|_C = \mathcal{O}_X(C)|_C \cong K_C$$ by adjunction.

$\Rightarrow$ i can be identified with the canonical map. [H. IV.]

1 Examples of K3 surfaces

A. Curves (Math 2036)

$g = 0$: $C \cong \mathbb{P}^1$

$g = 1$: C elliptic

$g = 2$: $C \xrightarrow[2:1]{} \mathbb{P}^1$ branched at 6 points.

$g = 3$: — hyperelliptic $C \xrightarrow[2:1]{} \mathbb{P}^1$

— quartic in $\mathbb{P}^2$ ($g = 3$ by adjunction)

$g = 4$: — hyperelliptic

— $C \cong Q \cap C \hookrightarrow \mathbb{P}^3$, Q = quadric, C = cubic.

$g = 5$: — hyperelliptic $C \xrightarrow[2-1]{} \mathbb{P}^1$

— trigonal $C \xrightarrow[3-1]{} \mathbb{P}^1$

— $C \cong Q_1 \cap Q_2 \cap Q_3 \hookrightarrow \mathbb{P}^4$, Q_i = quadrics.

Question Can this go on?

How about $g = 6, 7, 8, 9, \ldots$?

B. Analogous question.

Construct (or classify) low genus K3s.

If (X, L) is a pair, L (very) ample, $L^2 = 2g-2$ we say g = genus.

Features: $\boxed{a}$ we seek analogies with curves

we'd need analogues of

$g=1$ curves $\rightsquigarrow$ elliptic fibrations w/ sections

hyperelliptic curves $\rightsquigarrow$ hyperelliptic K3s

} later

general case $\longrightarrow$ Mukai examples for $g \leq 9$.

$\boxed{b}$ **Outcome:** we will learn about the moduli of K3s in low genera

Bonus: additional facts about low genus curves & 3-folds

Methods $\boxed{i}$ $g = 2$: double covers; $X \to \mathbb{P}^2$ branched along sextics

$\boxed{ii}$ $3 \le g \le 9$, various complete intersections

$\boxed{iii}$ Kummer surfaces $\rightsquigarrow$ abelian surfaces.

Math 206 - Lecture 3

January 13, 2020

0. Last time (X, L), $L^2 = 2g - 2$.

If L very ample $\Rightarrow$ $i : X \hookrightarrow \mathbb{P}^g$, $g + 1 = h^0(X, L)$

Smooth hyperplane section $C = X \cap H$ has genus g

Today & next time Construct examples in low genus

Method 1 : Complete Intersections

Methods 2 & 3 : - next time

I. $3 \le g \le 5$

Genus $g = 3$ $X \hookrightarrow \mathbb{P}^3$ smooth quartic,

$\mathcal{L} = \mathcal{O}_X(1) \Rightarrow \mathcal{L}^2 = 4 = 2g-2 \Rightarrow g = 3$. We check

[a] $K_X \cong \mathcal{O}_X$

[b] $H^1(X, \mathcal{O}_X) = 0$

[a] We use the normal sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^3}/_X \longrightarrow N_{X/\mathbb{P}^3} \cong \mathcal{O}_X(4) \longrightarrow 0.$$

Take determinants & dualize

$$K_{\mathbb{P}^3}/_X \cong K_X \otimes \mathcal{O}_X(-4) \cong \mathcal{O}_X \quad \text{since} \quad K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4).$$

Recall the calculation of $K_{\mathbb{P}^3}$. The Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0$$

Taking determinants & dualizing: $K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$.

[b] $H^1(X, \mathcal{O}_X) = 0$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$\Rightarrow \underbrace{H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})}_{0} \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \underbrace{H^2(\mathbb{P}^3, \mathcal{O}(-4)_{\mathbb{P}^3})}_{0.} \Rightarrow H^1(X, \mathcal{O}_X) = 0$$

Fact Over $\mathbb{P}^2$, line bundles have no intermediate cohomology

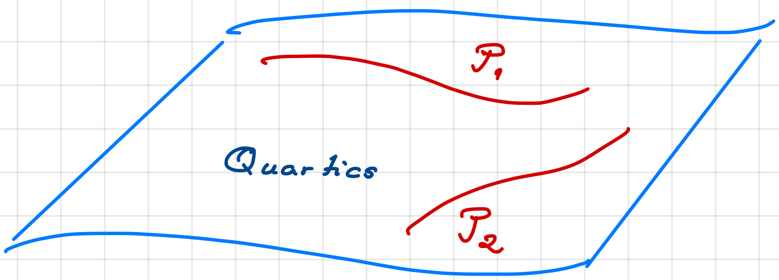
Count "moduli"

$$\dim \mathbb{P}H^0\underbrace{(\mathbb{P}^3, \mathcal{O}(4))}_{\binom{4+3}{3}-1} - \dim \underbrace{PGL_4}_{15} = 19.$$

(X, L), $L^2 = 4$, L ample

The moduli space $\mathcal{F}_3^\circ$ (not yet constructed in class)

K3's



$g = 3$ curves

$X \hookrightarrow \mathbb{P}^2$ quartics

$X \xrightarrow[2:1]{} \mathbb{P}^1$ hyperelliptic

$\mathcal{F}_3$ birational to $\mathbb{P}\, H^0(\mathbb{P}^3, \mathcal{O}(4)) / PGL_4$. $\rightsquigarrow$ unirational

$\mathbb{P} \dashrightarrow \mathcal{F}_3^\circ$ dominant

Question Describe the Betti numbers of $\mathcal{F}_3^\circ$.

Complete intersections $g = 4$, $g = 5$.

Setup

$$X \hookrightarrow \mathbb{P}^{r+2}, \quad X = Y_1 \cap \dots \cap Y_r$$

Y_i: degree d_i hypersurface, $\mathcal{L} = \mathcal{O}_X(1)$.

Canonical bundle $K_X \cong \mathcal{O}_X$. Use the normal sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^{r+2}}\big|_X \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X(d_i) \longrightarrow 0$$

Taking determinants & dualizing

$$K_X \cong \mathcal{O}_X\left(\sum_{i=1}^{r} d_i - r - 3\right). \cong \mathcal{O}_X$$

$$\Rightarrow \sum_{i=1}^{r} d_i = r+3. \qquad d_i > 1.$$

New examples

$\boxed{i}$ $r = 2$, $(d_1, d_2) = (2, 3)$

$$X \hookrightarrow \mathbb{P}^4, \quad X = Q \cap C$$

Q quadric, C cubic, $\deg X = 6 = 2g - 2 \Rightarrow g = 4$

$\boxed{ii}$ $r = 3$, $(d_1, d_2, d_3) = (2, 2, 2)$

$$X \hookrightarrow \mathbb{P}^5, \quad X = Q_1 \cap Q_2 \cap Q_3$$

$\deg X = 8 = 2g - 2 \Rightarrow g = 5$

Count "moduli" $(g=4)$

$$\underbrace{\dim \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^4}(2))}_{\text{choice of }[Q]} + \underbrace{\dim \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^4}(3))^*}_{[C]\text{ independent of }[Q]} - \underbrace{\dim PGL_5}$$

$$= \left(\binom{4+2}{2}-1\right) + \left(\binom{4+3}{3}-1-5\right) - 24$$

$$= 19$$

$H^1(X, \mathcal{O}_X) = 0$

Recall: over $\mathbb{P}^k$, line bundles have no intermediate cohomology (H. III. 5).

Remark X smooth projective, $\mathcal{F} \to X$ locally free is said to be arithmetically Cohen - Macaulay (ACM).

$$H^k(X, \mathcal{F}(p)) = 0 \quad \forall \quad 1 \leq k \leq \dim X - 1 \quad \forall p.$$

$\mathcal{O}_{\mathbb{P}^n}$ = ACM over $\mathbb{P}^n$ & $\mathcal{O}_{\mathbb{P}^n}(\ell)$ = ACM over $\mathbb{P}^n$.

Remark* (Horrocks) over $\mathbb{P}^m$, the only indecomposable vector bundles that are ACM are $\mathcal{O}_{\mathbb{P}^m}(\ell)$.

Lemma X complete intersection $\Rightarrow$ $\mathcal{O}_X$ is ACM.

Proof We argue by induction on $\dim X$. $X = \mathbb{P}^r$ done!

Let $Y = X \cap H$, H hypersurface of degree e

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(p-e) \longrightarrow \mathcal{O}_X(p) \longrightarrow \mathcal{O}_Y(p) \longrightarrow 0$$

Take cohomology

$$\underbrace{H^i(\mathcal{O}_X(p))}_{0} \longrightarrow \underbrace{H^i(\mathcal{O}_Y(p))}_{0} \longrightarrow \underbrace{H^{i+1}(\mathcal{O}_X(p-e))}_{0}$$

if $1 \leq i \leq \dim Y - 1$

From the Lemma we see $H^1(X, \mathcal{O}_X) = 0$. in the above examples.

II. $\underline{6 \le g \le 10}$ & $\underline{g = 12}$

__Idea__ Instead of projective space, use

Z smooth Fano i.e. K_Z^{-1} ample.

__Def__ Z is prime if $\mathrm{Pic}(Z) = \mathbb{Z}$.

__Index__ Write $K_Z^{-1} = M^{\otimes i}$, $i \in \mathbb{Z}_{>0}$, M ample & primitive.

i = index of Z.

__Coindex__ $c = \dim Z + 1 - i$

__Examples__ $\boxed{I}$ $Z = \mathbb{P}^{r+2}$, $K_{\mathbb{P}^{r+2}} = \mathcal{O}(-r-3)$

$\Rightarrow i = r+3 \Rightarrow c = (r+2) + 1 - i = 0.$

$\boxed{II}$ $Z = Q \hookrightarrow \mathbb{P}^{r+3}$ smooth quadric

$\Rightarrow K_Q = \mathcal{O}(-r-4+2) = \mathcal{O}(-r-2) \Rightarrow i = r+2$

$c = (r+2) + 1 - (r+2) = 1.$

$\boxed{III}$ $Z = G(2, \mathbb{C}^n) \rightsquigarrow K_Z.$

$\dim Z = 2(n-2)$

Over $Z = G(2, \mathbb{C}^n)$: $0 \longrightarrow E \longrightarrow \mathbb{C}^n \otimes \mathcal{O} \longrightarrow F \longrightarrow 0$ (1)

E, F tautological subbundle & quotient.

Recall $T_Z = Hom(E, F) = E^\vee \otimes F$.

Tensor by $E^\vee$:

$$0 \longrightarrow E \otimes E^\vee \longrightarrow \mathbb{C}^n \otimes E^\vee \longrightarrow \underbrace{F \otimes E^\vee}_{T_Z} \longrightarrow 0 .$$

Take determinants

$$\underbrace{\det (E \otimes E^\vee)}_{\text{trivial}} \otimes \det T_Z = (\det E^\vee)^{\otimes n}$$

$$\Rightarrow \quad K_Z \cong (\det E)^{\otimes n} \quad \Rightarrow \quad i = n .$$

$$c = \dim Z + 1 - i = 2(n-2) + 1 - n =$$

$$= n - 3 .$$

For instance $G(2,5)$ has $c = 2$, $G(2,6)$ has $c = 3$.

Exercise Coindex of $SG(k,n)$, $OG(k,n)$.*

Construction Z Fano as above, $\dim Z = r+2$, $K_Z^{-1} = M^{\otimes i}$

$$X = Z \cap H_1 \cap \dots \cap H_r \qquad M \text{ very ample}$$

$H_j \in |M^{\otimes d_j}|$. We hope to get a K3.

Canonical bundle $K_X \cong \mathcal{O}_X$.

$$0 \longrightarrow T_X \longrightarrow T_Z|_X \longrightarrow N_{X/Z} = \bigoplus_{j=1}^{r} M^{\otimes d_j} \longrightarrow 0$$

Take determinants & dualize

$$\Rightarrow \quad K_X \cong M^{\otimes\left(-i + \sum_{j=1}^{r} d_j\right)} \cong \mathcal{O}_X$$

We wish to have $\sum_{j=1}^{r} d_j = i$ index of Z

$$\Rightarrow \quad \operatorname{coindex}(Z) = (r+2)+1-i$$

$$= r+3-\sum_{j=1}^{r} d_j \leq 3.$$

Lemma $c \geq 0$

Proof $c \geq 0 \iff i \leq \dim Z + 1$.

Recall $K_Z^{-1} = M^{\otimes i}$. Consider

$f(t) = \chi(Z, M^{\otimes t}) =$ polynomial in t of degree $\dim Z$.

by Hirzebruch-Riemann-Roch.

Show: f has roots $-1, -2, \ldots, 1-i \Rightarrow i-1 \leq \dim Z$

This complete the proof. We show

- $h^k(Z, M^{\otimes t}) = 0 \quad \forall k \quad \forall t \in \{-1, \ldots, 1-i\}$

If $k=0$, $h^0(Z, M^{\otimes t}) = 0$ since $t<0$, M ample

$k>0$, $H^k(Z, M^{\otimes t}) = H^k(Z, K_Z \otimes M^{\otimes (i+t)}) = 0$

by Kodaira vanishing using $i+t>0$.

Discussion

Kobayashi – Ochiai (1973)

$\boxed{i}$ $r = 0 \implies Z = \mathbb{P}^{r+2}$

This case yielded the examples $3 \leq g \leq 5$.

Kobayashi – Ochiai

$\boxed{ii}$ $r = 1 \implies Z = Q \hookrightarrow \mathbb{P}^{r+3}$ quadric

Since $\sum_{j=1}^{r} d_j = r+2 \implies (d_1 \ldots d_r) = (1, 1, \ldots, 1, 3)$ & $(1, 1, \ldots 2, 2)$

Again this yields the old examples.

$\boxed{iii}$ $r = 2$. Since $\sum_{j=1}^{r} d_j = r+1 \implies (1, 1, \ldots, 1, 2)$

In this case $\dim Z = r+2$, $K_Z^{-1} = M^{\otimes(r+1)}$

These are called del Pezzo manifolds.

Indeed, these generalize del Pezzo surfaces.

If Z = del Pezzo surface = $Bl_k \mathbb{P}^2$, $\dim Z = 2$, $r = 0$

index $= 1$: $K_Z = -3H - \sum_i E_i$

We saw above that $G(2, \mathbb{C}^5)$ is another example.

$\boxed{IV}$ $r = 3$ $\Rightarrow \sum_{j=1}^{r} d_j = r \Rightarrow (1, 1, \dots, 1)$

In this case, $\dim Z = r + 2$, $K_Z^{-1} = M^{\otimes r}$

These are called Mukai manifolds.

We saw above that $G(2, \mathbb{C}^6)$ is an example.

Is $g = 0$? Yes. We only discuss $r = 3$; $r = 2$ is similar.

Lemma $\boxed{I}$ Z Fano of coindex $c = 3 \Rightarrow X = Z \cap H$, $H \in |M|$.

then X Fano of coindex $c = 3$ if $\dim Z \geq 4$

$\boxed{II}$ If $\dim Z = 3 \Rightarrow H^1(X, \mathcal{O}_X) = 0$.

Proof will discuss briefly next time.

Math 206 - Lecture 4

January 15, 2020

Plan

- Review from last time
- del Pezzo manifolds $r = 2$
- Mukai manifolds $r = 3$
- Conclusion

Last time — General construction

Z Fano manifold, $K_Z^{-1} = M^{\otimes i}$, i = index

$c = \dim Z + 1 - i$ coindex, $\dim Z = r+2$

$X = Z \cap H_1 \cap \ldots \cap H_r$, $H_i \in |M^{\otimes d_i}|$

To get $K_X \cong \mathcal{O}_X$ we only focus on

$c = 2$, $(d_1 \ldots d_r) = (1, 1, \ldots, 2)$

$c = 3$, $(d_1, \ldots, d_r) = (1, 1, \ldots, 1)$.

Questions

[i] How do we know we don't get X = abelian surface?

We show $H^1(X, \mathcal{O}_X) = 0$.

[ii] What is the genus $(X, M|_X)$?

We only discuss the case $c = 3$. The case $c = 2$ is similar.

Lemma [i] Z Fano of coindex $c=3$ $\Rightarrow$ $X = Z \cap H$, $H \in |M|$.

then X Fano of coindex $c=3$ if $\dim Z \geq 4$

[ii] If $\dim Z = 3$ $\Rightarrow$ $H^1(X, \mathcal{O}_X) = 0$.

Proof [i] Write $d = \dim Z$, $\dim X = d-1$, $d \geq 4$

The condition $\operatorname{coindex}(Z) = 3$ means $\operatorname{index}(Z) = d-2 \geq 2$.

Use adjunction

$$K_X = (K_Z + M)\big/_X = M^{\otimes(1-\operatorname{index}_Z)}\big/_X = M^{-(d-3)}\big/_X .$$

Since $d \geq 4$ $\Rightarrow$ X is Fano. We need to show $\operatorname{index}(X) = d-3$.

This amounts to showing $M/_X$ is primitive. When M is very ample, $d \geq 4$, the Grothendieck - Lefschetz thm shows

$\operatorname{Pic}(Z) \longrightarrow \operatorname{Pic}(X)$ is an isomorphism.

(very ampleness happens in all examples below).

$\operatorname{index}(X) = d-3$. $\Rightarrow$ $\operatorname{coindex}(X) = 3$.

Remark

For a different argument, use X, Z are both Fano.

In part [ii] below we show $H^1(Z, \mathcal{O}_Z) = 0$, $H^1(X, \mathcal{O}_X) = 0$ for all Fanos. This implies that the rows in the diagram below are injective

$$\begin{array}{ccccc} \mathrm{Pic}(Z) & \xrightarrow{\text{injective}} & H^2(Z, \mathbb{Q}) & & \text{b/c. } H^1(Z, \mathcal{O}_Z) = 0 \\ \downarrow & & \| & & \\ \mathrm{Pic}(X) & \xrightarrow{\text{injective}} & H^2(X, \mathbb{Q}) & & \text{b/c } H^1(X, \mathcal{O}_X) = 0 \end{array}$$

& by the weak Lefschetz thm $H^2(Z, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$.

Using the diagram: M primitive $\Rightarrow M|_X$ primitive.

[ii] $\quad 0 \longrightarrow \mathcal{O}_Z(-H) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_X \longrightarrow 0$

Take cohomology

$$H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^2(Z, M^{-1})$$

Since $\dim Z = 3$ & $r = 3 \Rightarrow \mathrm{index}(Z) = 1 \Rightarrow K_Z = M^{-1}$

Thus $H^1(Z, \mathcal{O}_Z) = H^1(Z, K_Z + M) = 0$ by Kodaira

$H^2(Z, M^{-1}) = H^1(Z, \mathcal{O}_Z)^\vee = 0$ by Serre duality.

$\Rightarrow H^1(X, \mathcal{O}_X) = 0.$

Genus. Since $(X, M/_X)$ is a K3 surface we can ask for the genus $2g-2 = (M/_X)^2$. Since $X = Z \cap H_1 \cap \dots \cap H_r$

(codim $X = 3$) $\Rightarrow 2g-2 = (M/_X)^2 = M^{r+2}$.

Remark

If Z is Fano 3-fold of codim 3 $\Rightarrow$ index $= 1$

$$\Rightarrow M = K_Z^{-1} \quad \Rightarrow 2g-2 = (-K_Z)^3$$

Define Genus $(Z) = \frac{1}{2}(-K_Z)^3 + 1.$

Coindex $c = 2$ (Iskovskikh, Fujita, '78 '80)

These are called del Pezzo manifolds.

Classification :

a) $\exists$ 4 nonprime examples in dim ≥ 3

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{P}^2 \times \mathbb{P}^2, \quad Bl_p \mathbb{P}^3, \quad \mathbb{P}\, T_{\mathbb{P}^2}$$

b) 5 prime examples in dim ≥ 3

i) 3-fold: degree 6 hypersurface in weighted projective space $w\mathbb{P}(1,1,1,2,3)$

ii) 2-sheeted covers of $\mathbb{P}^{r+2}$ branched over quartics

iii) smooth cubics in $\mathbb{P}^{r+3}$

iv) $(2,2)$ complete intersections in $\mathbb{P}^{r+4}$

v) $G(2, \mathbb{C}^5)$ & linear sections

Remark

(1) Examples $\boxed{a}$ yield K3s with Picard rank > 1. These are not generic, so we will not discuss them further.

(2) Examples $\boxed{I}$ – $\boxed{IV}$ yield $g \leq 5$.

Thus we only consider the last example. $G(2, \mathbb{C}^5)$.

K3's of genus $g = 6$ Let $Z = G(2, \mathbb{C}^5)$.

We saw last time coindex $G(2, \mathbb{C}^n) = n-3$.

Recall the Plücker embedding

$$G(k, \mathbb{C}^n) \xrightarrow{i} \mathbb{P} \Lambda^k \mathbb{C}^n$$

$$[W \subseteq \mathbb{C}^n] \longrightarrow [\Lambda^k W \subseteq \Lambda^k \mathbb{C}^n]$$

We have $\mathcal{O}_G(1) = i^* \mathcal{O}_{\mathbb{P}}(1) = \det \mathcal{S}^\vee$ where

$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_G \longrightarrow Q \longrightarrow 0$ is the

tautological sequence.

In our case

$$G(2, \mathbb{C}^5) \xrightarrow{i} \mathbb{P} \Lambda^2 \mathbb{C}^5 \cong \mathbb{P}^9$$

Since $\dim Z = 2(5-2) = 6$, we need

$X = Z \cap H_1 \cap H_2 \cap H_3 \cap Q$ where

Q quadric in $\mathbb{P}^9$ & H_i hyperplanes $\Rightarrow X = K3$.

This is consistent with $(d_1, \dots d_r) = (1, 1, \dots, 1, 2)$.

Note $Z \hookrightarrow \mathbb{P}^9 \Rightarrow X = Z \cap Q \cap H_1 \cap H_2 \cap H_3 \hookrightarrow \mathbb{P}^6$

We claim $(X, \mathcal{O}_X(1))$ has genus 6

Fact $G(k, \mathbb{C}^n) \xrightarrow{i} \mathbb{P}(\Lambda^k \mathbb{C}^n)$ has degree

$$(*) \quad (k(n-k))! \cdot \prod_{\substack{1 \le i \le k \\ k < j \le n}} (j-i)^{-1}.$$

When $k = 2$, the above specializes to Catalan number

$$\frac{1}{n-1} \binom{2n-4}{n-2}$$

In our case $\deg(Z \hookrightarrow \mathbb{P}^9) = 5 \Rightarrow \deg(X \hookrightarrow \mathbb{P}^6) = 10$

By definition $\deg(X) = 2g - 2 \Rightarrow 2g - 2 = 10 \Rightarrow g = 6.$

The fact above is classical, but see Borel & Hirzebruch (1958) for the case of general G/P.

One possible argument is to first compute the Hilbert series of the Plücker embedding:

$$\chi\left(G(k,\mathbb{C}^n),\ \mathcal{O}_G(1)^{\otimes N}\right) = \prod_{\substack{1\le i\le k\\ k<j\le n}} \frac{N+j-i}{j-i}$$

This follows by interpreting $H^0\left(G(k,\mathbb{C}^n),\ \mathcal{O}_G(1)^{\otimes N}\right)$ as a GL_n-representation & using Weyl dimension formula.

Expanding into powers of N:

$$\chi\left(G(k,\mathbb{C}^n),\ \mathcal{O}_G(1)^{\otimes N}\right) \sim \frac{N^{\dim G}}{(\dim G)!}\cdot \text{degree} + \dots$$

$$\prod_{\substack{1\le i\le k\\ k<j\le n}} \frac{N+j-i}{j-i} \sim N^{\dim G}\cdot \prod_{\substack{1\le i\le k\\ k<j\le n}} (j-i)^{-1} + \dots$$

we get the claim.

Coindex 3 Mukai (88) & Massimiliano Mola (99)

Z Mukai manifold

$K_Z^{-1} = M^r$, $\dim Z = r+2$, $d_1 = \dots = d_r = 1$.

Remark

$\exists$ non-prime examples

$\mathbb{P}^3 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^2 \times Q^3$, ...

Prime examples

$\boxed{i}$ $Z = G(2, \mathbb{C}^6) \Rightarrow g = 8$

$\boxed{ii}$ $Z = OG(5, \mathbb{C}^{10}) \Rightarrow g = 7$

$\boxed{iii}$ $Z = LG(3, \mathbb{C}^6) \Rightarrow g = 9$

$\boxed{iv}$ $Z = G_2/P \Rightarrow g = 10$

$\boxed{v}$ $Z = SG(3, \mathbb{C}^7, \omega_1, \omega_2, \omega_3) \Rightarrow g = 12$.

These & linear sections are all prime examples if $g \geq 7$.

Remark (Borel & Hirzebruch)

All G/P's are Fano manifolds.

In the cases considered here this can be checked by hand.

Remark W prime Fano 3-fold, $c = 3$, $\text{genus}(W) = (-K_W)^3/2 + 1$

Thus $g \le 10$ & $g = 12$. as the above list shows.

Remark We obtain examples of curves $7 \le g \le 10$, $g = 12$ as well by intersecting with sufficiently many hyperplanes.

"Old" example $g = 8$, $Z = G(2, \mathbb{C}^6)$, coindex $2 = 6 - 3 = 3$.

Exercise $Z = G(2, \mathbb{C}^6) \hookrightarrow \mathbb{P}\Lambda^2\mathbb{C}^6 \cong \mathbb{P}^{14}$ degree 14

(by previous argument). $\dim Z = 2 \cdot 4 = 8$

$X = Z \cap H_1 \cap \dots \cap H_6 \hookrightarrow \mathbb{P}^8$ is a K3 surface

$2g - 2 = \text{degree}(X) = \text{degree}(Z) = 14 \Rightarrow g = 8.$

Discussion of the remaining examples

(1) $Z = LG(3,6)$

Description of $LG(n,2n)$ Take $V \cong \mathbb{C}^{2n}$

Let $\omega = e_1^\vee \wedge e_2^\vee + \dots + e_{2n-1}^\vee \wedge e_{2n}^\vee$ symplectic form.

Let

$$LG(n,2n) = \left\{ W \subseteq V,\ \dim W = n,\ \omega\big|_{W\times W} \equiv 0 \right\}$$

Note $LG(n,2n) \hookrightarrow G(n,2n)$.

Over $G(n,2n)$ there is the universal subbundle

$$\mathcal{S} \hookrightarrow V \otimes \mathcal{O}_G.$$

The form

$$\omega : \Lambda^2 V \longrightarrow \mathbb{C} \text{ induces by restriction}$$

$$\tilde{\omega} : \Lambda^2 \mathcal{S} \longrightarrow \mathcal{O}_G \quad \& \quad LG = \text{Zero}(\tilde{\omega}).$$

$$\dim LG(n,2n) = \dim G(n,2n) - \operatorname{rank} \Lambda^2 \mathcal{S}$$

$$= n^2 - \binom{n}{2} = \frac{n(n+1)}{2}$$

Canonical bundle & coindex

normal bundle

$$0 \longrightarrow T_{LG} \longrightarrow T_G/_{LG} \longrightarrow \Lambda^2 \mathcal{S}^\vee \longrightarrow 0$$

Taking determinants we see:

$$K_G/_{LG} = K_{LG} \otimes \det \Lambda^2 \mathcal{S}^\vee$$

We showed last time $K_G \cong \mathcal{O}_G(-2n)$.

We have $\det \Lambda^2 \mathcal{S}^\vee = (\det \mathcal{S}^\vee)^{n-1} = \mathcal{O}_G(n-1)$

$\Rightarrow K_{LG} \cong \mathcal{O}_{LG}(-n-1) \Rightarrow \operatorname{index}(LG) = n+1.$

$\Rightarrow \operatorname{coindex}(LG) = 1 + \frac{n(n+1)}{2} - (n+1) = 3 \Leftrightarrow n = 3.$

Plücker embedding

Note $\dim LG(3,6) = 6.$, $LG(3,6) \hookrightarrow G(3,6) \hookrightarrow \mathbb{P}\Lambda^3 \mathbb{C}^6 \cong \mathbb{P}^{19}$

Let $\omega^\# : \Lambda^3 V \longrightarrow V$, $V \cong \mathbb{C}^6$

$$a \wedge b \wedge c \longrightarrow \omega(a,b)\, c + \omega(b,c)\, a + \omega(c,a)\, b.$$

Note $\omega^\#$ surjective $\Rightarrow \dim \operatorname{Ker} \omega^\# = \binom{6}{3} - 6 = 14.$

Note $\omega/_{W\times W} \equiv 0 \iff \omega^{\#}/_{\wedge^3 W} = 0$. Thus the Plücker point $\wedge^3 W \hookrightarrow \wedge^3 V$ lies in $\operatorname{Ker} \omega^{\#}$ showing that .

$$LG(3,6) \hookrightarrow \mathbb{P}^{13}.$$

The K3 surface of genus 9

$\Rightarrow$ $X = LG(3,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \hookrightarrow \mathbb{P}^9$ is a K3 surface. The fact below shows

$$\deg X = \deg LG(3,6) = 16 = 2g-2 \Rightarrow g = 9.$$

Fact The degree of Plücker embedding of $LG(n,2n)$

$$2^d d! \prod_{1\le i\le j\le n} (2n+2-i-j)^{-1}, \quad d = \dim LG(n,2n)$$

(Borel & Hirzebruch 1958)

(2) $g = 12$, Z = "trisymplectic Grassmannian."

Definition Let $\omega_1, \omega_2, \omega_3$ be general symplectic forms

$$Z = SG_3(3, \mathbb{C}^7) = \left\{ W \subseteq \mathbb{C}^7,\ \dim W = 3,\ \omega_i\big|_{W \times W} \equiv 0 \right\}$$

Dimension

We have $Z = \text{zero}(\tilde{\omega}_1) \cap \text{zero}(\tilde{\omega}_2) \cap \text{zero}(\tilde{\omega}_3)$

where $\tilde{\omega}_i$ are sections of $\Lambda^2 \mathcal{S}^\vee$, $\mathcal{S} \longrightarrow G(3, \mathbb{C}^7)$.

$$\Rightarrow \dim Z = \dim G(3, \mathbb{C}^7) - 3 \operatorname{rank} \Lambda^2 \mathcal{S}^\vee$$

$$= 3(7-3) - 3 \cdot \binom{3}{2} = 3.$$

Coindex

$$0 \longrightarrow T_Z \longrightarrow T_{G(3,7)}\big/_Z \longrightarrow \bigoplus_{i=1}^{3} \Lambda^2 \mathcal{S}^\vee\big/_Z \longrightarrow 0$$

We have $K_{G(3,7)} \cong \mathcal{O}_G(-7)$ & $\det \Lambda^2 \mathcal{S}^\vee \cong (\det \mathcal{S}^\vee)^{\otimes 2} \cong \mathcal{O}_G(2)$

$\Rightarrow K_Z = \mathcal{O}_Z(-1) \Rightarrow \text{index } Z = 1 \Rightarrow \text{coindex}(Z) = 3.$

Plücker embedding.

We have $G(3, \mathbb{C}^7) \hookrightarrow \mathbb{P}(\wedge^3 \mathbb{C}^7)$. but the Plücker points of W's in $SG_3(3, \mathbb{C}^7)$ lie in

$$\mathbb{P}(\operatorname{Ker} \omega_1^\# \cap \operatorname{Ker} \omega_2^\# \cap \operatorname{Ker} \omega_3^\#) \cong \mathbb{P}^{13}.$$

The K3 surface of genus 12

Then $X = Z \cap H \hookrightarrow \mathbb{P}^{12}$ is a K3 surface of genus 12.

(3) Z = G_2 - variety, $g = 10$

The group G_2

A "legal" definition of G_2 is as follows. Let $V \cong \mathbb{C}^7$

Pick a basis $e_1, \dots, e_7$ of $V^\vee$ and write

$$e_{ijkl} = e_i^\vee \wedge e_j^\vee \wedge e_k^\vee \wedge e_l^\vee = 4\text{-form}$$

Let

$$\varphi = -e_{4567} + e_{2367} - e_{2345} + e_{1357} + e_{1346} + e_{1256} - e_{1247}$$

The group G_2 is the stabilizer of φ in $GL(V)$.

It can be checked $\dim G_2 = 14$.

Cylinders of Calabi-Yau geometries

Over $\mathbb{R}$, the group G_2 can be understood as follows

Let

$$\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3 = \text{cylinder over } \mathbb{C}^3$$

Note $\mathbb{C}^3$ is Calabi-Yau with

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3 = \text{Calabi-Yau form.}$$

$$\omega = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$$ Kähler form.

Let t be the coordinate on $\mathbb{R}$. Then

$$\psi = -dt \wedge \operatorname{Im} \Omega - \frac{1}{2} \omega \wedge \omega.$$

when written in real coordinates (t, x_i), $dz_i = dx_{2i-1} + \sqrt{-1}\, dx_{2i}$

<u>Back to complex world</u> Alternatively, we can pick

ψ to be a generic form in $\Lambda^4 V^\vee$, $V \cong \mathbb{C}^7$.

$\dim \Lambda^4 V^\vee = \binom{7}{4} = 35$. $\dim GL(V) = 49$.

Generic means the $GL(V)$ orbit of ψ in $\Lambda^4 V^\vee$ is open.

Then $G_2 = \operatorname{Stab}_{GL(V)} \psi = \dim 49 - 35 = 14$.

<u>The G_2 - variety</u>

Define $Z = \{ W \subseteq V, \dim W = 5, \psi|_{W \times W \times W \times W} \equiv 0 \}$.

$Z \hookrightarrow G(5, \mathbb{C}^7)$ is cut out by a section of $\Lambda^4 \mathcal{S}^\vee$.

$\Rightarrow \dim Z = \dim G(5, \mathbb{C}^7) - \operatorname{rk} \Lambda^4 S^\vee$

$$= 5(7-5) - \binom{5}{4} = 5.$$

Coindex

From $0 \longrightarrow T_Z \longrightarrow T_{G(5,\mathbb{C}^7)}\big/_Z \longrightarrow \Lambda^4 S^\vee\big/_Z \longrightarrow 0$ we find

$$K_Z \cong \mathcal{O}_Z(-3) \Rightarrow \text{index } Z = 3 \Rightarrow \text{coindex } Z = 3.$$

Plücker

Note $Z \hookrightarrow G(5, \mathbb{C}^7) \hookrightarrow \mathbb{P}\Lambda^5\mathbb{C}^7 \cong \mathbb{P}^{20}$.

Let $\varphi^\#: \Lambda^5 V \longrightarrow V$ is the contraction with φ

$$W \text{ is in } Z \Longleftrightarrow \varphi^\#\big/_{\Lambda^5 W} \equiv 0.$$

The Plücker point $\Lambda^5 W$ of W lies in $\mathbb{P}\operatorname{Ker} \varphi^\#$

$$\dim \operatorname{Ker} \varphi^\# = \dim \Lambda^5 V - \dim V = \binom{7}{5} - 7 = 14.$$

$\Rightarrow Z \hookrightarrow \mathbb{P}^{13}.$

K3 surface of genus 10

Finally $X = Z \cap H_1 \cap H_2 \cap H_3 \hookrightarrow \mathbb{P}^{10}$ is a K3 surface of genus 10.

Math 206 - Lecture 5

January 20, 2020

Lecture Notes in Canvas, "Files"

Plan

- genus $g = 7$

- conclusion & remarks $3 \leq g \leq 10$, $g \neq 12$

Summary of last time

$Z = G/P$ Fano of coindex 3, $\dim Z = r+2$.

$\boxed{I}$ $\quad Z = G(2, \mathbb{C}^6) \implies g = 8$

$\boxed{II}$ $\quad Z = LG(3, \mathbb{C}^6) \implies g = 9$

$\boxed{III}$ $\quad Z = SG(3, \mathbb{C}^7, \omega_1, \omega_2, \omega_3) \implies g = 12.$

$\boxed{IV}$ $\quad Z = G_2(5, \mathbb{C}^7, \varphi) \implies g = 10$

$\boxed{V}$ $\quad Z = OG(5, \mathbb{C}^{10}) \implies g = 7$

Write $K_Z = M^{-r}$, M primitive, $H_i \in |M|$

K3 surface	curve	Fano 3-fold,
$X = Z \cap H_1 \cap \dots \cap H_r$	$C = Z \cap H_1 \cap \dots \cap H_{r+1}$	$M = Z \cap H_1 \cap \dots \cap H_{r-1}$

These examples were discussed last time except

$$Z = OG(5,10) \quad \text{or} \quad Z = OG(n, \mathbb{C}^{2n})$$

Definition Q nondegenerate quadratic form on $V \cong \mathbb{C}^{2n}$

The orthogonal grassmanian is

$$OG(n, 2n) = \{ W : \dim W = n, \ W \subseteq V, \ Q|_W \equiv 0 \}.$$

Notation Consider the quadric:

$$Y_Q = \{ z \in \mathbb{P} V : \ Q(z) = 0 \} \hookrightarrow \mathbb{P}^{2n-1}$$

Note $Q|_W \equiv 0 \implies \underbrace{\mathbb{P} W}_{(n-1) \dim l} \subseteq Y_Q.$

Model In local coordinates, we will take

$$Q = z_1 z_{2n} + z_2 z_{2n-1} + \dots + z_n z_{n+1}$$

$$Q = \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & & 1 & \vdots \\ & \iddots & & \vdots \\ 1 & \cdots & & 0 \end{pmatrix}$$

$W \subseteq \mathbb{C}^{2n}$, $\dim W = n$

$$W = \operatorname{span}(w_1, \dots, w_n) \rightsquigarrow W = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix}$$

We will not distinguish between the subspace & matrix.

$$W \text{ isotropic for } Q \iff W^t Q W = 0$$

Question Why is $OG(n,2n)$ subtler?

Reason #1 $OG(n,2n)$ has 2 components.

Example $n=1$: $Q = zw$ on $\mathbb{C}^2$

$$OG(1,2) = \langle e_1 \rangle \text{ and } \langle e_2 \rangle.$$

sign change from previous page

Example $n=2$: $Q = xw - yz$ in $\mathbb{C}^4$

$$Y_Q = \{ xw - yz = 0 \} \hookrightarrow \mathbb{P}^3.$$

We have $Y_Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ via Segre embedding

$$[a:b], [c:d] \longrightarrow [ac : bc : ad : bd]$$

Note Y_Q has 2 rulings, yielding two components

$$OG(2,4) = \mathbb{P}^1 \sqcup \mathbb{P}^1.$$

Underlying reason — 2 orbits for $SO(2n, \mathbb{C})$ action.

If $V = \langle e_1, \dots e_{2n} \rangle$, let

$$W = \langle e_1, \dots e_n \rangle, \quad W' = \langle e_1, \dots e_{n-1}, e_{n+1} \rangle$$

Check

[I] W, W' are in different orbits.

[II] W, W' are in the same orbit if

$$\dim (W \cap W') \equiv n \mod 2$$

Thus $OG(n, 2n)$ has 2 components $OG^+(n, 2n)$ & $OG^-(n, 2n)$.

These are isomorphic. We will just pick one of these components.

Convention $V = F + F^\vee$, $\dim F = n$, F fixed & isotropic

$$OG^+ = \{ W : \dim (W \cap F) \equiv n \mod 2 \}.$$

Canonical bundle & coindex

$\mathcal{S} \to G(n,2n)$ subbundle , $\mathcal{S}/_w = [w]$

We have

$$OG(n,2n) \hookrightarrow G(n,2n) \hookrightarrow \mathbb{P}(\wedge^n \mathbb{C}^{2n})$$

Note $\mathcal{O}_{\mathbb{P}}(1)$ restricts to $\det \mathcal{S}^\vee$ on $G = G(n, \mathbb{C}^{2n})$

$$Q : \operatorname{Sym}^2 \mathbb{C}^{2n} \longrightarrow \mathbb{C} \rightsquigarrow \tilde{Q} : \operatorname{Sym}^2 \mathcal{S} \longrightarrow \mathcal{O}_G .$$

$OG = \operatorname{Zero}(\tilde{Q})$. , $\tilde{Q}$ section of $\operatorname{Sym}^2 \mathcal{S}^\vee$

$$\dim OG = \dim G(n,2n) - \operatorname{rank} \operatorname{Sym}^2 \mathcal{S}^\vee = n^2 - \binom{n+1}{2} = \frac{n(n-1)}{2}.$$

As last time (see the computation for LG):

$$K_{OG^\pm} = K_G\big/_{OG^\pm} \otimes \det \operatorname{Sym}^2 \mathcal{S}^\vee\big/_{OG} =$$

$$= (\det \mathcal{S})^{2n}\big/_{OG^\pm} \otimes \left(\det \mathcal{S}\big/_{OG^\pm}\right)^{-(n+1)}$$

$$= \left(\det \mathcal{S}^\vee\big/_{OG^\pm}\right)^{-(n-1)}$$

It appears that $\operatorname{index}(OG^\pm) = n-1$. This is not right.

Issue $\det \mathcal{S}\big/_{OG^\pm}$ is not primitive but rather

2 - divisible

Reason #2

$\det \mathcal{J}/_{OG^\pm}$ is not primitive in $\mathrm{Pic}(OG^\pm)$

Example $n=2$ We have shown $OG(2,4) = \mathbb{P}^1 \sqcup \mathbb{P}^1$

If we pick one such $\mathbb{P}^1$, consider Plücker embedding

$$OG^\pm(2,4) \cong \mathbb{P}^1 \hookrightarrow G(2,4) = \text{quadric in } \mathbb{P}^5.$$

One can check that

$$\mathcal{O}_{\mathbb{P}^5}(1)/_{OG^\pm} = \det \mathcal{J}^\vee/_{OG^\pm} = \mathcal{O}_{\mathbb{P}^1}(2)$$

Indeed each $[a:b] \in \mathbb{P}^1$ yields

$$W = \langle a e_1 + b e_2,\ a e_3 + b e_4 \rangle \text{ isotropic}$$

$$\Rightarrow \Lambda^2 W = \text{span } (a e_1 + b e_2) \wedge (a e_3 + b e_4)$$

$$= a^2\, e_1 \wedge e_3 + b^2\, e_2 \wedge e_4 + ab\,(e_1 \wedge e_4 + e_3 \wedge e_2)$$

$\Rightarrow$ Plucker coordinates are quadratic in a & b.

Question Why was $\det \mathcal{J}$ primitive for G or LG?

Recall $G(k, \mathbb{C}^n) \longrightarrow \mathbb{P} \wedge^k \mathbb{C}^n$ Plücker

$$\det \mathcal{J} = \mathcal{O}_{\mathbb{P}}(1)\big/_{G(k,n)}.$$

Claim $\exists\ \Sigma$ curve in $G(k,n)$ with

$$\deg \det \mathcal{S}^\vee\big/_{\Sigma} = 1 \implies \det \mathcal{J} \text{ primitive.}$$

Proof Define

$$\Sigma = \left\{ W = \langle e_1, \dots, e_{k-1}, te_k + se_{k+1} \rangle : [t,s] \in \mathbb{P}^1 \right\}$$

is a curve in $G(k,n)$. Then $\Sigma \cong \mathbb{P}^1$ and

$$\det \mathcal{S}\big/_{\Sigma} \cong \mathcal{O}_{\Sigma}(-1) \implies \text{degree} \det \mathcal{S}^\vee\big/_{\Sigma} = 1.$$

Remark

Note that Σ can be described as

$$\widetilde{W} + W_0 \quad \text{where} \quad \widetilde{W} = \langle e_1 \dots e_{k-1} \rangle$$

and $W_0 = \langle te_k + se_{k+1} \rangle$ = varying line.

Aside

(1) the same argument works for $LG(n,2n)$

$$\Sigma = \{ W = W_0 + \tilde{W} : \tilde{W} \in LG(n-1, 2n-4) \text{ fixed}$$

$$\& \; W_0 \text{ varies in } LG(1,2) = G(1,2) = \mathbb{P}^1 \}$$

(2) In the case of $OG^{\pm}(n,2n)$

$$\Sigma = \{ W = W_0 + \tilde{W} : \tilde{W} \in OG^{+}(n-2, 2n-4) \text{ fixed}$$

$$W_0 \text{ varies in } OG^{+}(2,4) \cong \mathbb{P}^1 \}$$

gives a degree 2 curve because of previous example.

Question What is the primitive generator?

$$OG^{\pm}(n, 2n) \hookrightarrow \mathbb{P} S^{\pm}$$ spinor embedding

We will see that $\mathcal{O}_{\mathbb{P}S^{\pm}}(1)/_{OG^{\pm}}$ is primitive and

$$\det \mathcal{T}^{\vee} \cong \mathcal{O}_{\mathbb{P}S^{\pm}}(2)/_{OG^{\pm}}$$

Construction $W \subseteq \mathbb{C}^{2n}$ isotropic for Q. Represent W by a matrix & row reduce it to $W = \begin{pmatrix} I \\ u \end{pmatrix}$.

$$W^t Q W = \begin{pmatrix} I & u^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I \\ u \end{pmatrix} = u + u^t = 0$$

$\Rightarrow u$ skew symmetric

Recall Plücker embedding

$$G(n, 2n) \longrightarrow \mathbb{P} \wedge^n \mathbb{C}^{2n}, \quad W \longrightarrow \wedge^n W$$

$u \longrightarrow$ all $n \times n$ minors of W

$=$ all $j \times j$ minors of u, $0 \leq j \leq n$

Crucial Remark A skew symmetric, $A + A^t = 0$

- A has odd dimension $\Rightarrow \det A = 0$

- A has even dimension $2k \times 2k$

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \Rightarrow \det A = a^2, \quad Pf(A) = a \Rightarrow \boxed{\det A = Pf(A)^2.}$$

Define
$$Pf(A) = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} (-1)^\sigma a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2k-1)\,\sigma(2k)}$$

Better If $A = (a_{ij})$, define

$$\omega = \sum_{i<j} a_{ij}\, e_i \wedge e_j \Rightarrow \boxed{\frac{1}{k!}\omega^k = Pf(A)\, e_1 \wedge \dots \wedge e_{2k}}$$

Indeed $\omega \wedge \dots \wedge \omega$ is a $2k$-form so it is proportional to $e_1 \wedge \dots \wedge e_{2k}$.

Important fact

$$\mathrm{Pf}(A)^2 = \det A$$

Proof $A = C^T \begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_n \\ & & & \lambda_n & 0 \end{pmatrix} C$ & compute using

$$\det(C^t B C) = (\det C)^2 \det B$$

$$\mathrm{Pf}(C^t B C) = \det C \cdot \mathrm{Pf}(B)$$

Spinor embedding

$$OG^{\pm}(n, 2n) \xhookrightarrow{i} \mathbb{P}S^{\pm}$$

$\mathcal{U} \longrightarrow$ All Pffafians of principal $2j \times 2j$ minors of $\mathcal{U}$.

$$\dim S^{\pm} = 2^{n-1}$$

A more canonical definition requires pure spinors.

$\underline{n=2}$ This construction yields $OG^{\pm}(2,4) \cong \mathbb{P}^1$.

Why is $OG \overset{i}{\hookrightarrow} \mathbb{P}S$, $\mathcal{O}_{\mathbb{P}S}(1)/_{OG}$ primitive?

Use the same curve

$$\Sigma = \left\{ W = W_0 + \tilde{W} : \begin{array}{l} W_0 \in OG^{+}(2,4) \text{ varying} \\ \tilde{W} \in OG^{+}(n-2, 2n-4) \text{ fixed} \end{array} \right\}$$

$\underline{\text{Back}}$ $\underline{\text{to}}$ $\underline{OG^{\pm}(n,2n)}$

det $J/_{OG}$ is 2-divisible

We see index $= 2(n-1)$

$$\Rightarrow \text{coindex} = 1 + \frac{n(n-1)}{2} - 2(n-1) = 3 \iff n = 5.$$

Outcome $g = 7$

$\dim OG^{\pm}(5,10) = 10$. The spinor embedding

$$Z = OG^{\pm}(5,10) \hookrightarrow \mathbb{P}^{15} \text{ has degree } 12.$$

$$X = Z \cap H_1 \cap \dots \cap H_8 \hookrightarrow \mathbb{P}^7$$

$$C = Z \cap H_1 \cap \quad \cap H_9 \text{ is a genus 7 curve}$$

Remark We can also use $X \hookrightarrow G(2, \mathbb{C}^5)$ cut out by a section of $\mathcal{O}_G(1) + \mathcal{O}_G(1) + J^{\vee}(1)$. to describe $g = 7$ K3s. But the above picture involving $OG^{\pm}$ is more uniform.

Conclusions

Remark The above constructions show $\mathcal{F}_g$ unirational $3 \le g \le 10$ & $g = 12$. (with more work).

$\mathcal{F}$ unirational if $\exists\, N$ $\mathbb{A}^N \dashrightarrow \mathcal{F}$ dominant

Example $g = 8$: $\quad Z = G(2, \mathbb{C}^6) \hookrightarrow \mathbb{P}\Lambda^2\mathbb{C}^6 = \mathbb{P}^{14}$

$$X = Z \cap H_1 \cap \dots \cap H_6 \,, \quad H_i \in |\mathcal{O}_{\mathbb{P}^{14}}(1)|$$

To determine X we need to pick a point in

$$\mathrm{Grass}(6, H^0(Z, \mathcal{O}_Z(1)))$$

and divide by PGL_6. Thus

$$\mathrm{Grass}(6, H^0(Z, \mathcal{O}_Z(1)))/PGL_6 \dashrightarrow \mathcal{F}_8.$$

Count moduli: $\dim \mathrm{Grass} = 6 \cdot (15 - 6)$

$$\dim PGL_6 = 35$$

$$\dim \mathcal{F}_8 = 19.$$

Thus we expect the map to be dominant (it is) &

thus $\mathcal{F}_8$ is dominated by a rational variety $\mathrm{Gross}(6, \mathbb{C}^{15})$

hence it is unirational.

<u>How about</u> genus 8 <u>curves?</u>

Intersect one more time with a hyperplane

$$C = Z \cap H_1 \cap \dots \cap H_7.$$

$$\mathrm{Grass}(7, H^0(Z, \mathcal{O}_Z(1)))/PGL_6 \dashrightarrow \mathcal{M}_8$$

count moduli & match : $\dim \mathrm{Grass} = 7(15-7)$

$\dim PGL_6 = 35$

Match! $\quad \dim \mathcal{M}_8 = 3.8-3.$

This argument eventually shows $\mathcal{M}_8$ is unirational.

Unirationality of $\mathcal{F}_g$ is established by these methods for

$3 \leq g \leq 10$ & $g = 12$.

What about other values of g?

- $g = 11, 13, 16, 18, 20$ Mukai by pushing the above descriptions
- $g = 14, 22$ Farkas – Verra
- $g = 45, 51, 53, 55, 58, 59, 61, g > 62$, $\mathcal{F}_g$ general type

Hulek – Gritschenko – Sankaran (2007)

using number theory / representation theory

Question Find a proof of $\mathcal{F}_g$ general type using only algebraic geometry.

Compare this with $\mathcal{M}_g$.

- $g \leq 10$ Severi proved M_g is unirational (1915)

He conjectured M_g unirational for all g, but this turned out false.

The K3 methods recover Severi's result for $g \leq 9$.

What goes wrong for us when $g = 10$? (later).

- $g = 11, 13$ Chang - Ran
  $g = 12$ Sernesi
  $g = 14$ Verra
  $\Rightarrow$ M_g unirational for these values of g.

- $g = 16$ Chang - Ran claimed the same, but the proof does not hold.

Aside – Severi's idea (works for $g \le 10$)

$$\mathcal{N} = \{ C \hookrightarrow \mathbb{P}^2 ;\ \delta\text{-nodal, geometric genus } g \}$$

$$\begin{tikzcd} & \mathcal{N} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ M_g & & \underbrace{\mathrm{Hilb}_\delta \mathbb{P}^2}_{\text{rational}} \end{tikzcd}
\qquad
\begin{tikzcd} & C \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \tilde{C} & & \mathrm{Sing}(C) \end{tikzcd}$$

Show:

- $\mathrm{Hilb}_\delta \mathbb{P}^2$ rational
- $\mathcal{N}$ is birational to a projective bundle over $\mathrm{Hilb}_\delta \mathbb{P}^2$ so rational
- π dominant

$\Downarrow$

M_g unirational

The case $g = 10$ Why don't the above methods show

$\mathcal{M}_{10}$ is rational? (But show $\mathcal{F}_{10}$ is rational.)

Recall $Z \hookrightarrow \mathbb{P}^{13}$, $\dim Z = 5$, $Z = G_2$ - variety

$$X = Z \cap H_1 \cap H_2 \cap H_3$$

$$C = Z \cap H_1 \cap H_2 \cap H_3 \cap H_4$$

where H_i are hyperplanes in $\mathbb{P}^{13}$.

count moduli: $\underbrace{\mathrm{Grass}(3, \mathbb{C}^{14})}\,/\underbrace{G_2} \dashrightarrow \underbrace{\mathcal{F}_{10}}$

$$\dim \quad 3.(14-3) \;-\; 14 \;=\; 19$$

For curves $\underbrace{\mathrm{Grass}(4, \mathbb{C}^{14})}\,/G_2 \dashrightarrow \mathcal{M}_{10}$

$$\dim = 4\,(14-4) - 14 = 26$$

$$\dim \mathcal{M}_{10} = 3g - 3 = 27.$$

This is the only case where the dimension count fails.

Thus a generic curve of genus $g=10$ does not lie on a K3 surface.

$$\Longrightarrow \quad \text{Divisor of curves on K3's} \hookrightarrow M_{10}$$

26 27

Studied by Cukierman (1989) & Farkas – Popa (2004)

to disprove the slope conjecture of Morrison - Harris.

<u>Final</u> <u>Remark</u> Calabi – Yau 3 - folds

$$h^1(Y, \mathcal{O}_Y) = h^2(Y, \mathcal{O}_Y) = 0$$

$$K_Y \cong \mathcal{O}_Y$$

are connected with Fano varieties of coindex 4.

$\rightsquigarrow$ classification due to Carla Novelli

However we already have many examples of CY^3's.

Math 220B — Lecture 6

January 22, 2021

Plan

- double covers & Riemann - Hurwitz

- genus $g=2$ K3 surfaces

- elliptic K3 surfaces.

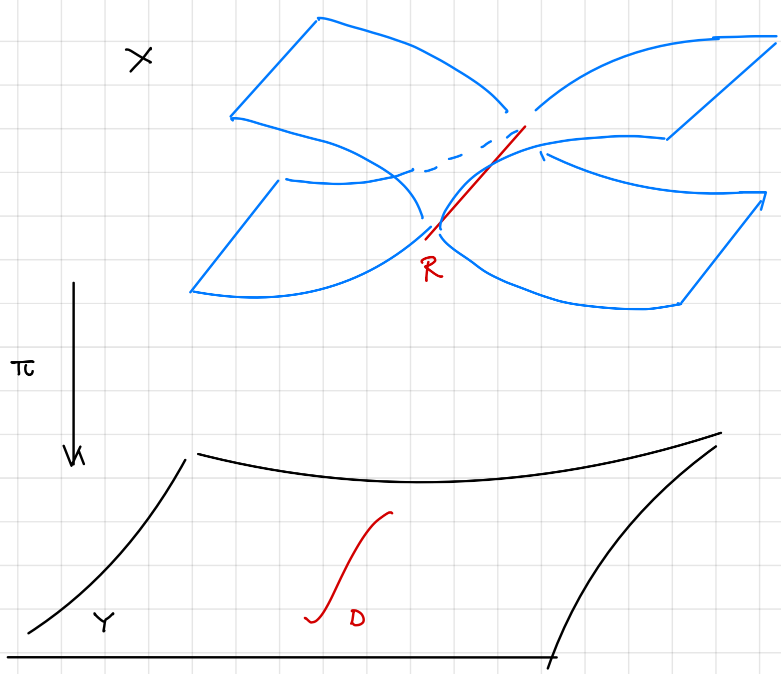
Rahul's Lecture on January 29, 10:30–12

Method 1 — Complete intersections in Fano manifolds $3 \le g \le 10$

Method 2 — Cyclic (branched) covers

- Y smooth
- $D \xhookrightarrow{s=0} Y$ smooth divisor
- $\mathcal{L} \longrightarrow Y$, $\mathcal{L}^{\otimes n} \cong \mathcal{O}_Y(D)$

This data determines $X \xrightarrow[n:1]{} Y$ branched along D.



Plan $\boxed{\text{i}}$ give construction

$\boxed{\text{ii}}$ compute $H^1(X, \mathcal{O}_X)$ & canonical K_X

Outcome Do we get a K3 surface?

$\boxed{\text{iii}}$ analyze examples

$\boxed{\text{iv}}$ formulate some questions.

Construction $\underline{\mathcal{L}}$ = total space of $\mathcal{L}$, $p : \underline{\mathcal{L}} \longrightarrow Y$

$\boxed{a}$ $p^* \mathcal{L}$ has a canonical section t

This is clear set-theoretically.

Indeed, points of $\underline{\mathcal{L}}$ are pairs (y, ℓ), $\ell \in \mathcal{L}_y$.

The section

$$t\,(y, \ell) = \ell \in \mathcal{L}_y = (p^* \mathcal{L})_{(y,\ell)}.$$

Scheme-theoretically

section $p^* \mathcal{L} \iff \mathcal{O}_{\underline{\mathcal{L}}} \longrightarrow p^* \mathcal{L}$

$\iff p^* \mathcal{L}^\vee \longrightarrow \mathcal{O}_{\underline{\mathcal{L}}}$ (dualize)

$\iff \mathcal{L}^{-1} \longrightarrow p_* \mathcal{O}_{\underline{\mathcal{L}}}$ (adjoint functors)

There is a canonical choice since

$$p_* \mathcal{O}_{\underline{\mathcal{L}}} = \operatorname{Sym}^\bullet \mathcal{L}^\vee = \bigoplus_j \mathcal{L}^{-j}$$

[b] Define $X \subseteq \underline{\mathcal{L}}$ by the vanishing

$p^*s - t^n = 0$ as a section of $p^*\mathcal{L}^{\otimes n} \to \underline{\mathcal{L}}$

Note $\pi : X \hookrightarrow \underline{\mathcal{L}}$ is $n:1$ & branched at $s = 0$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \underline{\mathcal{L}} \arrow[d] \\ & Y \end{tikzcd}$$

i.e. π branched along D.

Local coordinates

$Y \hookleftarrow V \hookrightarrow \mathbb{C}^m$ coordinates $(y_1, \dots, y_m)$

$L|_V \cong V \times \mathbb{C}$ coordinates $(y_1 \dots y_m, t)$

Let $D = \{y_1 = 0\}$ in V

$p^* L \longrightarrow V \times \mathbb{C}$ is trivial with section

$$t(y_1 \dots y_m, t) = t$$

$$\begin{array}{ccccc} X & \cong & \pi^{-1}(V) & \subseteq & \underbrace{V \times \mathbb{C}}_{(y,\, t)} \\ \downarrow \pi & & \downarrow & & \\ Y & \supseteq & V & & \end{array}$$

given by $\{y_1 = t^n\} \iff p^* s = t^{\otimes n}$

SUMMARY

$$\begin{array}{ccc} X & \supseteq & \pi^{-1}(V) \\ \downarrow \pi & & \downarrow \\ Y & \supseteq & V \end{array}$$

$\pi^{-1}(V)$ has coordinates $(t, y_2, \dots, y_m)$

V has coordinates $(y_1, y_2 \dots y_m) = (t^n, y_2, \dots, y_m$

$y_1 = t^n$

Lemma $\pi_* \mathcal{O}_X = \mathcal{O}_Y + \mathcal{L}^{-1} + \dots + \mathcal{L}^{-(n-1)}$

Proof We give a proof in the analytic category. Take $V \subseteq Y$ sufficiently small coordinate chart, $\mathcal{L}/_V \cong V \times \mathbb{C}$.

To each $f \in (\pi_* \mathcal{O}_X)(V) = \mathcal{O}_X(\pi^{-1} V)$ we show how to associate $(a_0, \dots, a_{n-1})$ sections of $\mathcal{O}, \mathcal{L}^{-1}, \dots, \mathcal{L}^{-(n-1)}$ over V, in a manner compatible with restrictions.

Extend f to $\tilde{f}$ in $\mathcal{L}/_V \cong V \times \mathbb{C}$ & Taylor expand in a small polydisc

$$\tilde{f} = \sum_{k=0}^{\infty} t^k p^* A_k \,, \quad A_k \in \Gamma(V, \mathcal{L}^{-k})$$

Using $t^n = p^* s$ on X, we rewrite

$$f = \tilde{f}/_X = \sum_{k=0}^{n-1} t^k p^* a_k \,, \quad a_k \in \Gamma(V, \mathcal{L}^{-k}).$$

The association $f \longrightarrow (a_0, \dots a_{n-1})$ is the isomorphism of the Lemma.

Take $n=2 \Rightarrow \pi_* \mathcal{O}_X = \mathcal{O}_Y + \mathcal{L}^{-1}$.

$$H^1(X, \mathcal{O}_X) \underset{\pi \text{ finite}}{=} H^1(Y, \pi_* \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) + H^1(Y, \mathcal{L}^{-1}).$$

__Corollary__ $H^1(X, \mathcal{O}_X) = 0$ provided.

$$H^1(Y, \mathcal{O}_Y) = H^1(Y, \mathcal{L}^{-1}) = 0.$$

What about the canonical bundle?

__Hurwitz formula__ $K_X = \pi^*(K_Y \otimes \mathcal{L}^{n-1})$

__Take__ __$n=2$__ $K_X \cong \pi^*(K_Y \otimes \mathcal{L}) \cong \mathcal{O}_X$

__Conclusion__ X is $K3$ surface iff Y surface

(1) $H^1(Y, \mathcal{O}_Y) = 0$

(2) $D \in |-2K_Y|$ smooth

Indeed, take $\mathcal{L} = K_Y^{-1}$. Use Serre duality to see $H^1(Y, \mathcal{L}^{-1}) = 0$

Proof of Hurwitz

$n=2$: $\quad K_X = \pi^*(K_Y \otimes \mathcal{L})$.

$$\begin{array}{ccl} X \supseteq \pi^{-1}(V) & & \text{has coordinates } (t, y_2, \ldots, y_m) \\ \downarrow \pi \quad \downarrow & & \\ Y \supseteq V & & \text{has coordinates } (y_1, y_2 \ldots y_m) = (t^2, y_2, \ldots, y_m) \end{array}$$

Let $R = \frac{1}{2}\pi^* D$. Indeed $\pi^* D$ is not reduced since

$\pi^* y_1 = t^2$. $\Rightarrow R = \{t=0\} \Rightarrow \mathcal{O}_Y(R) = \pi^*\mathcal{L}$. We show

$$K_X - R = \pi^* K_Y.$$

Note K_Y is spanned by $dy_1 \wedge \ldots \wedge dy_m$

$\pi^* K_Y$ is spanned by $d\pi^* y_1 \wedge \ldots \wedge dy_m =$

$$= 2t \; dt \wedge dy_2 \wedge \ldots \wedge dy_m$$

K_X is spanned by $dt \wedge dy_2 \wedge \ldots \wedge dy_m$

$\mathcal{O}_Y(-R)$ is spanned by t

$\Rightarrow \pi^* K_Y = K_X - R$ as needed.

Example ($g=2$)

Take $Y = \mathbb{P}^2$, $K_Y \cong \mathcal{O}(-3) \Rightarrow D \in |\mathcal{O}_{\mathbb{P}^2}(6)|$

Pick D a smooth sextic & construct

$$X \xrightarrow{\pi} \mathbb{P}^2 \text{ double cover branched along } D.$$

$\Rightarrow X$ is K3 surface.

Let $\mathcal{L} = \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \Rightarrow \mathcal{L}^2 = 2 = 2g-2 \Rightarrow g = 2.$

Count moduli: $\dim \mathbb{P} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)) - \dim PGL_3 = 19.$

Silly Question

What happens if $g = 1$? In this case,

$(X, \mathcal{L})$, $\mathcal{L}^2 = 2g - 2 = 0$

Example $X \longrightarrow \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(f)$.

Genus of smooth fiber:

$$2\,\text{genus} - 2 = \mathcal{L}^2 + \mathcal{L}.K = 0 \Rightarrow \text{genus} = 1.$$

We will in fact show later that

$\mathcal{L} \neq \mathcal{O}$, $\mathcal{L}^2 = 0 \Rightarrow X$ is elliptic fibration

Beware $\mathcal{L}$ may not be the class of a fiber

Question Examples?

What other surfaces can we try?

$\boxed{\text{i}}$ $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $D \in (4,4)$ smooth curve

$X \xrightarrow{\pi} Y$ double cover branched along D is K3 surface

Two divisors $D_1 = \pi^*(\mathbb{P}^1 \times pt) \in \mathrm{Pic}(X)$

$D_2 = \pi^*(pt \times \mathbb{P}^1) \in \mathrm{Pic}(X)$

$\Rightarrow$ $D_1^2 = 0$, $D_2^2 = 0$, $D_1 . D_2 = 2$

Thus $\Lambda = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \hookrightarrow \mathrm{Pic}(X)$ so this is a special K3.

(Noether - Lefschetz locus)

A very interesting situation occurs for Hirzebruch surfaces.

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow \mathbb{P}^1$$

$n = 0 \implies \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \implies$ last example.

<u>Example</u> (Elliptic surfaces) $n = 4$

$Y = \mathbb{F}_4$ Hirzebruch surface.

<u>Exercise</u> $H^1(Y, \mathcal{O}_Y) = 0$

$D \in |-2K_Y|. \implies X \xrightarrow[2:1]{\pi} Y$ is $K3$ surface

We will check X is a $K3$ surface. & say a bit more.

First — A short discussion of Hirzebruch surfaces.

$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n))$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$$\begin{array}{c} \mathbb{F}_n \\ \downarrow \pi \\ \mathbb{P}^1 \end{array}$$

<u>Remark</u> $\mathbb{P}(V) \cong \mathbb{P}(V \otimes \mathcal{L})$.

Thus $\mathbb{F}_n \cong \mathbb{F}_{-n}$ taking $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n)$ & $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-n)$.

Take $n > 0$.

<u>Fact</u> (H. Chp $\underline{V}$.1)

$\mathbb{P}(\mathcal{V}) \longrightarrow C$ ruled surface

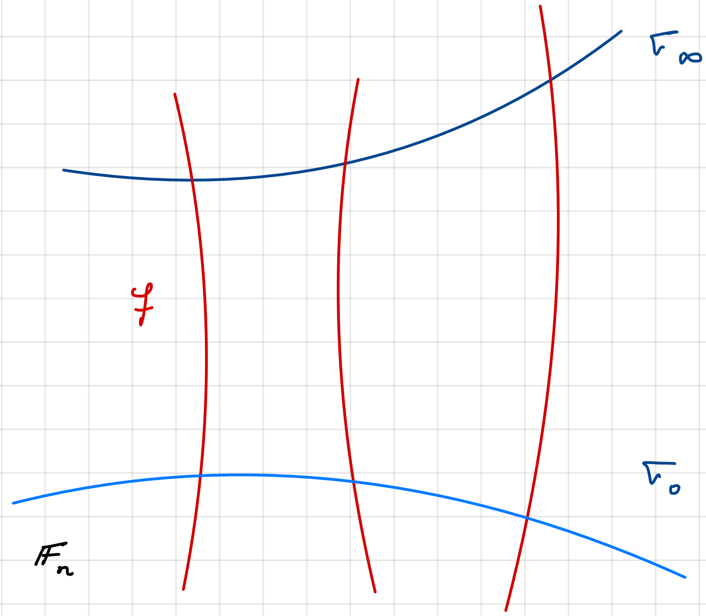
$$\operatorname{Pic} \mathbb{P}(\mathcal{V}) = \pi^* \operatorname{Pic}(C) + \mathbb{Z}$$

In our case $\operatorname{Pic}(\mathbb{F}_n) = \mathbb{Z} + \mathbb{Z}$.

Curve classes.

[i] fiber f, $f^2 = 0$

[ii] sections Γ_0, Γ_∞



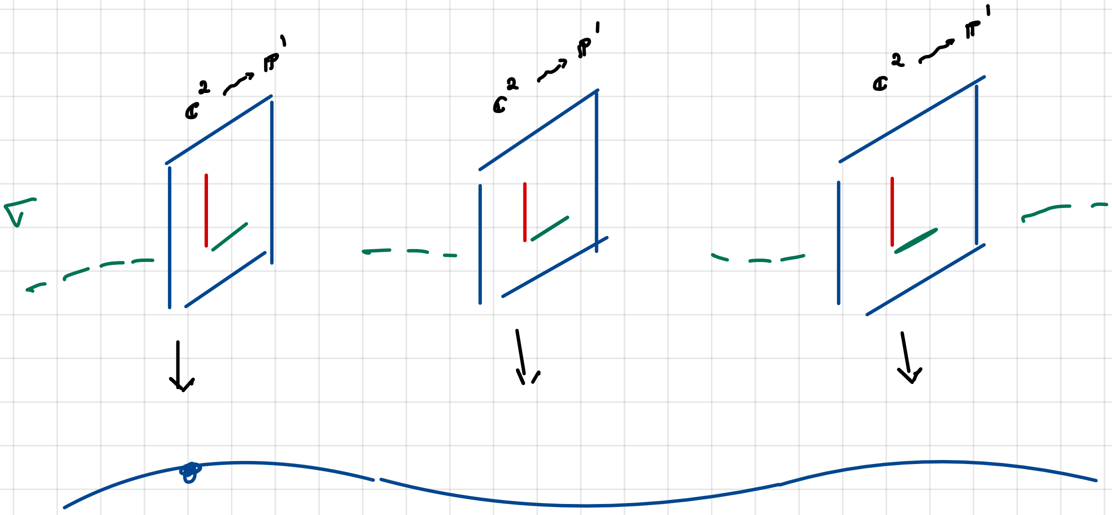
We have $\Gamma_0^2 = n$, $\Gamma_\infty^2 = -n$. (next page)

$$\Gamma_0 . f = \Gamma_\infty . f = 1 \implies \Gamma_\infty - \Gamma_0 = af.$$

Since $\Gamma_0^2 = n$, $\Gamma_\infty^2 = -n \implies a = -n \implies \Gamma_0 = \Gamma_\infty + nf$

$\implies$ $\Gamma_0 . \Gamma_\infty = 0$. since

$$\Gamma_0 . \Gamma_\infty = \Gamma_\infty (\Gamma_\infty + nf) = -n + n = 0.$$



$$\begin{array}{c} \mathbb{P}(\mathcal{O}_c + \mathcal{L}) \\ \downarrow \pi \;\uparrow \sigma \\ C \cong \mathbb{P}^1 \end{array}$$

In general if $\mathcal{V} = \mathcal{O}_C + \mathcal{L} \longrightarrow C$, $\mathbb{P}(\mathcal{V}) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} C$

Let σ correspond to $C = \mathbb{P}(\mathcal{O}_C + \underline{0}) \hookrightarrow \mathbb{P}(\mathcal{O}_C + \underline{\underline{\mathcal{L}}})$

$$N_{\sigma/\mathbb{P}\mathcal{V}} \cong \mathcal{L}$$

$$\sigma^2 = \deg \mathcal{L}$$

In our case, $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n))$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(n)$ gives the section σ_0 with $\sigma_0^2 = n$.

Similarly, $\mathbb{F}_n \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-n))$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-n)$ gives the section σ_∞ with $\sigma_\infty^2 = -n$

Claim $K_{\mathbb{F}} = -2\sigma_\infty - (n+2)f$

Write $K_{\mathbb{F}} = a\sigma_\infty + bf.$

Adjunction for $\sigma_\infty \cong \mathbb{P}^1$ & $f \cong \mathbb{P}^1$ gives

$$-2 = 2\,\text{genus}(\sigma_\infty) - 2 = \sigma_\infty^2 + K_{\mathbb{F}}.\sigma_\infty$$

$$= \sigma_\infty^2 + (a\sigma_\infty + bf)\sigma_\infty$$

$$= -n - an + b$$

$$-2 = 2\,\text{genus}(f) - 2 = f^2 + K_{\mathbb{F}}.f$$

$$= f^2 + (a\sigma_\infty + bf)f$$

$$= a$$

Solving we find $a = -2,\ b = -n-2.$

Linear series on Hirzebruch surfaces

Lemma [i] $\sigma_\infty + mf$ base point free iff $m \geq n$

[ii] $\sigma_\infty + mf$ very ample iff $m > n$.

Proof $D = \sigma_\infty + mf \Rightarrow D \cdot \sigma_\infty = m - n$

D bpf $\Rightarrow D \cdot \sigma_\infty \geq 0 \Rightarrow m \geq n$

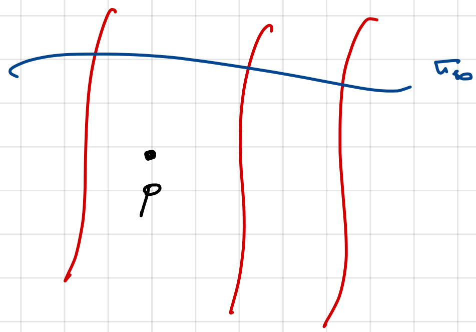
D very ample $\Rightarrow D \cdot \sigma_\infty > 0 \Rightarrow m > n$.

Conversely

[i] Let $p \in \mathbb{F}_n$. Wish to find $C \in |\sigma_\infty + mf|$ with $p \notin C$. We will pick the curve C to be either $\sigma_\infty + m$ fibers or $\sigma_0 + (m-n)$ fibers depending on $p \notin \sigma_\infty$ or $p \in \sigma_\infty$ (and hence $p \notin \sigma_0$). The fibers are chosen not to pass through p.

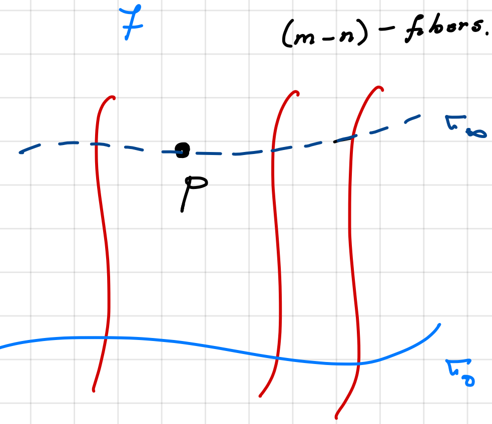
Proof by picture We have two cases:

$p \notin \Gamma_\infty$



$C = \Gamma_\infty + m$ fibers not through p

$p \in \Gamma_\infty$



$C = \Gamma_0 + (m-n)$ fibers

$C \in |\Gamma_\infty + mf|$

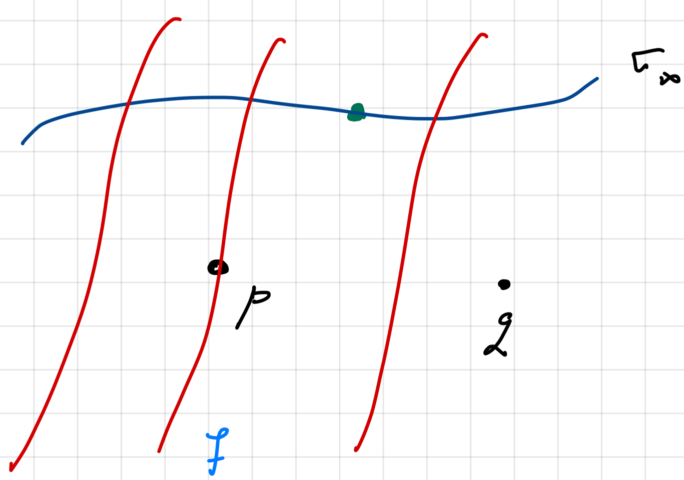
---

iii We wish to separate points & tangent vectors.

Let $p, q \in \mathbb{F}_n$, $p \neq q$. (or point p & tangent vector t)

Proof by picture Very similar to 1st case. (H. $\underline{\overline{V}}$)

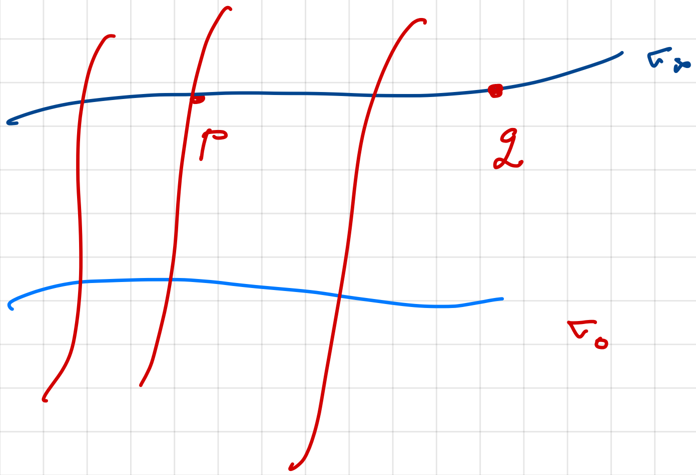
- p, q not in the same fiber & not both in Γ_∞.



$C = \Gamma_\infty + m$ fibers

one through p, none through q

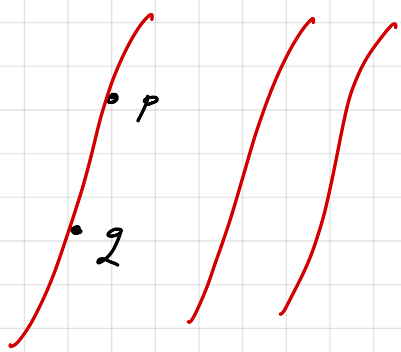
• $p, q \in \sigma_\infty$



$C = \sigma_0 + (n-m)$ fibers

one through p, none through q

• $p, q \in$ same fiber $D = \sigma_\infty + mf$



Conclude using (1) & (2):

(1) $\mathcal{O}_{\mathbb{F}}(D)\big/_f = \mathcal{O}_f(1)$ is very ample on f so separates p & q.

(2) $H^0(\mathcal{O}_{\mathbb{F}}(D)) \overset{\text{surjective}}{\twoheadrightarrow} H^0(\mathcal{O}_{\mathbb{F}}(D)\big/_f)$. Suffices to check

$H^1(\mathcal{O}_{\mathbb{F}}(D-f)) = 0$. Note

$$\pi_* \mathcal{O}_{\mathbb{F}}(D-f) = \pi_* \mathcal{O}(\sigma_\infty + (m-1)f) = \pi_* \mathcal{O}(\sigma_\infty) \otimes \mathcal{O}_{\mathbb{P}^1}(m-1)$$

$$= (\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(n)) \otimes \mathcal{O}_{\mathbb{P}^1}(m-1)$$

which has no H^1!

Remark

if $m \geq n \Rightarrow |\sigma_\infty + mf|$ contains smooth irred curve

Why? If

$\boxed{i}$ $m > n$ – because of Bertini & very ampleness

$\boxed{ii}$ $m = n$ – take σ_0.

Remark

Take $n > 0$, $a \neq 0$, $b \neq 0$ (H. Chp $\underline{V}$).

$\boxed{i}$ $a\sigma_\infty + bf$ very ample $\Longleftrightarrow b > an > 0$

Why? „$\Leftarrow$"

$$a\sigma_\infty + bf = (a-1)\underbrace{(\sigma_\infty + nf)}_{bpf} + \underbrace{(\sigma_\infty + \mu f)}_{\text{very ample}}, \quad \mu > n$$

$\boxed{ii}$ $|a\sigma_\infty + bf|$ contains smooth irred curve $\Longleftrightarrow b \geq an > 0$

Why? „$\Leftarrow$" Bertini + ε if $b = an$

The argument for $b = an$

$a(\Gamma_\infty + nf)$ contains a smooth irred curve

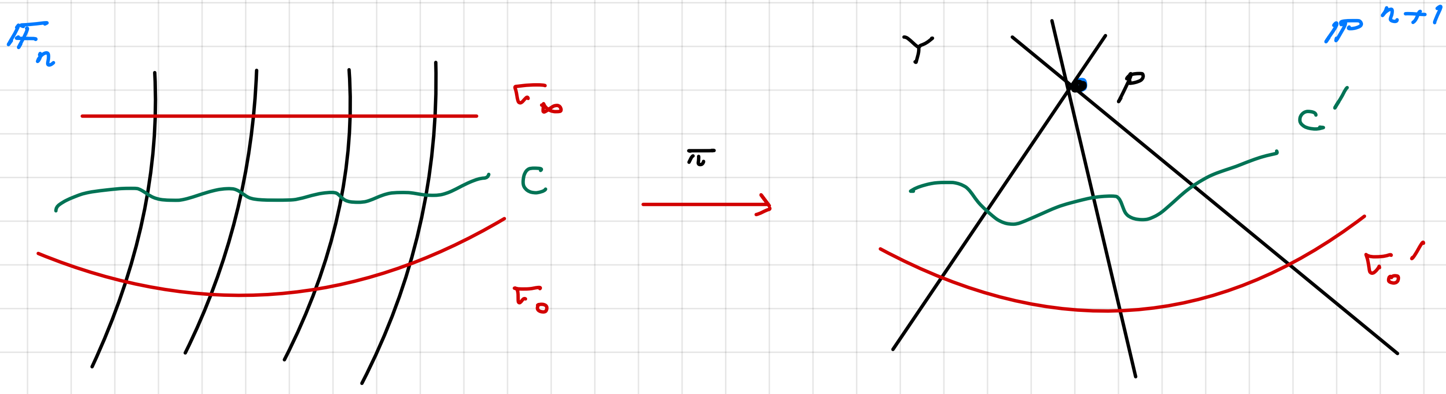
Let $\mathcal{L} = \Gamma_\infty + nf$, which is basepoint free. It induces

$$|\mathcal{L}| : \mathbb{F}_n \xrightarrow{\pi} \mathbb{P}^{n+1}$$

Check

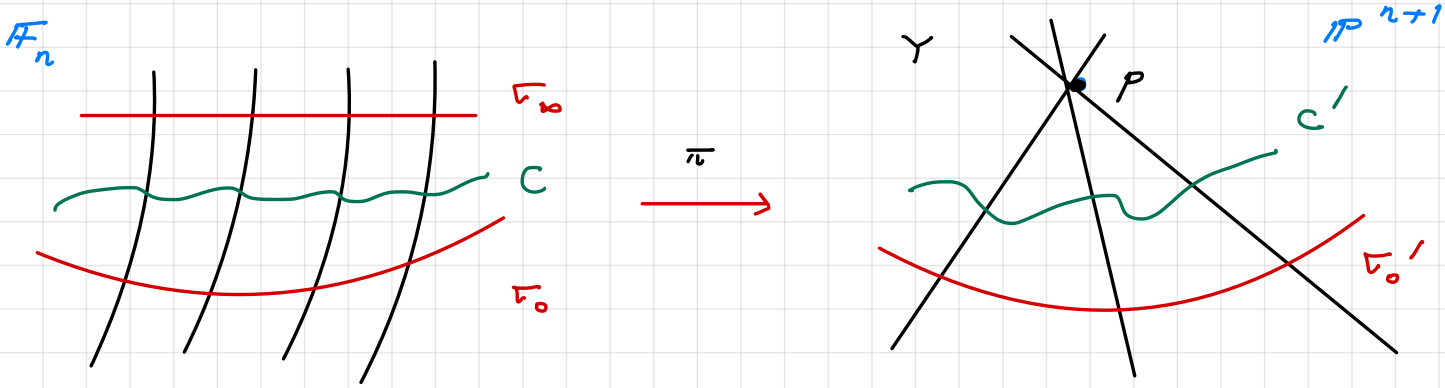
$$\chi(\mathcal{L}) = \chi(\mathcal{O}) + \frac{\mathcal{L}(\mathcal{L} - K_{\mathbb{F}})}{2} = n+2.$$

$\Gamma_\infty . \mathcal{L} = 0 \Rightarrow \Gamma_\infty$ contracted to p



$$\Gamma_0' = \pi(\Gamma_0)$$

π is birational



<u>Verify</u>

(1) Γ_0' has degree $L^2 = n$

(2) Γ_0' is hyperplane section

take $n+1$ points in Γ_0',
span a hyperplane H
Γ_0' is contained in H
by degree reasons

$\Gamma_0' = Y \cap H$

(3) $C' \in |aH|$, $p \notin C'$

smooth & irreducible possible
by Bertini

(4) $C = \pi^{-1}(C')$ is smooth

& irreducible

(5) $C \in |L^{\otimes a}|$ is a smooth & irred curve

The case $n = 4$

$X \xrightarrow[2:1]{\pi} \mathbb{F}_4$ double cover branched at D.

We need

$$D \in |-2K_{\mathbb{F}_4}| = |4\sigma_\infty + 12f|$$

We don't expect smooth & irreducible D.

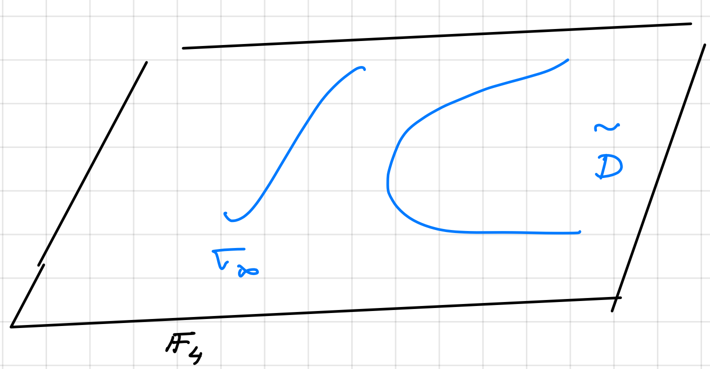
However, we can take D of the form

$$D = \sigma_\infty + \tilde{D}$$

possible by the above

$\tilde{D} \in |3\sigma_\infty + 12f|$ smooth & irreducible (see above)

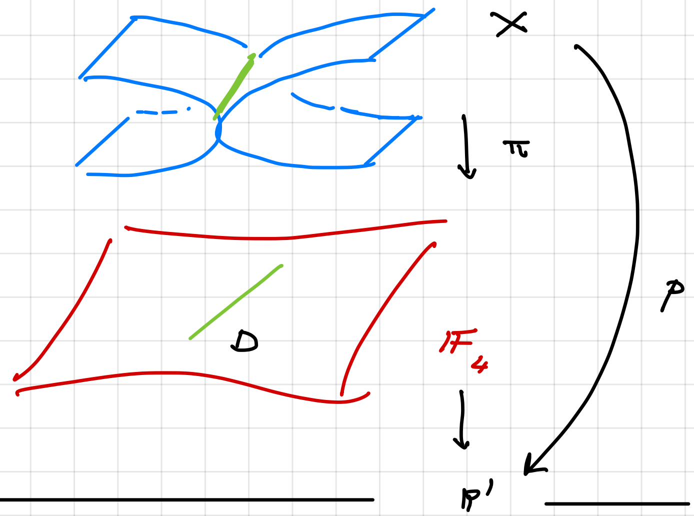
$\sigma_\infty . \tilde{D} = \sigma_\infty(3\sigma_\infty + 12f) = 0 \Rightarrow \sigma_\infty, \tilde{D}$ disjoint



The K3 surface

$D = \Gamma_\infty + \tilde{D}, \quad \tilde{D} \in |3\Gamma_\infty + 12 f|.$

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "p"'] & \mathbb{F}_4 \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$



$F = \pi^* f$

$S = \frac{1}{2}\pi^* \Gamma_\infty$

$\Longrightarrow$

$F^2 = 0$ fiber

$F \cdot S = 1$ section

$S^2 = -2$ (uses $n = 4$)

$S \cong \mathbb{P}^1$

In fact,

- $X \overset{F}{\underset{S}{\rightleftarrows}} \mathbb{P}^1$ is elliptic fibration

- F = fiber, S = section.

Note

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \hookrightarrow \mathrm{Pic}(X).$$

Conclusion

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \hookrightarrow \mathrm{Pic}(X).$$

(1) we constructed elliptic K3's with sections as double covers of $\mathbb{F}_4$

(2) $\exists$ moduli space $\mathcal{F}_\Lambda$

$\mathcal{F}_\Lambda$ is unirational. (Miranda 1981)

rational (Lejarraga, 1993).

These questions make sense for arbitrary lattices.

$$\Lambda \hookrightarrow \mathrm{Pic}(X).$$

($\exists$ moduli space $\mathcal{F}_\Lambda$ of Λ- polarized K3's which we will discuss later).

Question

(1) Is $\mathcal{F}_\Lambda$ unirational?

(2) Study the topology of $\mathcal{F}_\Lambda$?

$\boxed{a}$ $\Lambda = \langle 2g-2 \rangle \rightsquigarrow$ Mukai & Hulek–Gritsenko–Sankaran

$\boxed{b}$ $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \rightsquigarrow$ see above

$\boxed{c}$ $\Lambda_k = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2k \end{pmatrix}$ general type for $k \geq 220$

unirational for $k \leq 10$ + ...

Fortuna – Mezzedimi (2020)

$\boxed{d}$ high rank lattices. $\rightsquigarrow$ Dolgachev.

Plan

- short discussion of topology of K3s

- linear series on K3s

  (1) $\mathcal{L}^2 > 0$ & Reider techniques

  (2) $\mathcal{L}^2 = 0 \Rightarrow$ elliptic surfaces

- moduli of K3s constructed.

Math 220B - Lecture 7

January 27, 2021

Plan

- general discussion of topology of K3s
- some ideas that go in the constuction of the moduli space
- linear series on K3s

  (1) $\mathcal{L}^2 > 0$ & Reider techniques

  (2) $\mathcal{L}^2 = 0 \implies$ elliptic surfaces

Instead of the usual lecture

Rahul's AG Seminar on Friday

(Time has changed to 11:30 – 1).

Topological invariants of K3 surfaces

Fact All K3 surfaces are homeomorphic.

In light of this, the results below should not be surprising.

I. Old Facts

Recall (Lecture 2)

$$e_{top}(X) = 24 \quad \text{because} \quad \chi(X, \mathcal{O}_X) = \frac{K_X^2 + e(X)}{12}.$$

$\Longrightarrow$ $b_0(X) = 1$

$b_1(X) = 0$

$b_2(X) = 22$

$b_3(X) = 0$

$b_4(X) = 1$

Example 1 $X \hookrightarrow \mathbb{P}^3$ quartic surface

$$e_{top}(X) = \int_X c_2(T_X) \qquad \text{Poincaré – Hopf / Gauss – Bonnet}$$

$$0 \to T_X \to T_{\mathbb{P}^3}|_X \to \mathcal{O}_X(4) \to 0$$

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes \mathbb{C}^4 \to T_{\mathbb{P}^3} \to 0 .$$

$$c(T_X) = \frac{c(T_{\mathbb{P}^3}|_X)}{c(\mathcal{O}(4)|_X)} = \frac{c(\mathcal{O}_X(1))^4}{c(\mathcal{O}_X(4))} = \frac{(1+H|_X)^4}{1+4H|_X} =$$

$$= (1+4H|_X + 6H^2|_X + \dots)(1-4H|_X + 16H^2|_X - \dots)$$

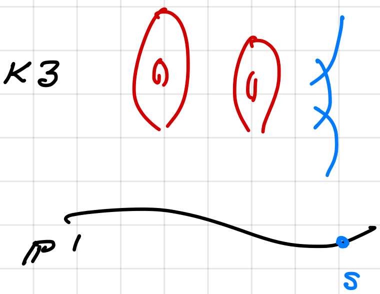
$$= 1 + 6H^2|_X$$

$$c_2(T_X) = 6H^2|_X = 24 \quad \text{since}$$

$$X \text{ quartic} \Rightarrow H^2|_X = 4.$$

Example 2

$X \longrightarrow \mathbb{P}^1$ elliptic fibration, $X = K3$



$S \subseteq \mathbb{P}^1$ singular fibers

$X \setminus \pi^{-1}S \longrightarrow \mathbb{P}^1 \setminus S$ topologically locally trivial

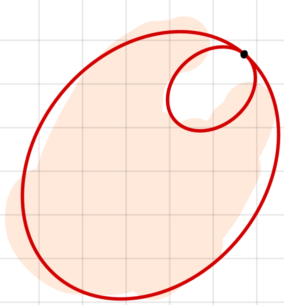
fibers $e_{top} = 0$ (Ehresmann)

$$e_{top}(X) = \underbrace{e_{top}(X \setminus \pi^{-1}S)} + \sum_{s \in S} e(X_s)$$

$$= 0 + \sum_{s \in S} e(X_s) = 24.$$

If X_s are reduced & irreducible $\Rightarrow p_a(X_s) = 1$.

X_s $\qquad \Rightarrow e(X_s) = 1$



Conclusion: 24 singular fibers.

Aside Adjunction formula for singular curves

C smooth $\Rightarrow$ $2g-2 = C^2$. adjunction

$g = 1 + \frac{C^2}{2}$.

If $C \hookrightarrow X$ reduced & irreducible

$p_a(C) = \dim h^1(C, \mathcal{O}_C)$ = arithmetic genus.

$$\Rightarrow \quad p_a(C) = 1 + \frac{C^2}{2}. \quad \geq 0$$

Proof Use the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)).$$

$$= \chi(\mathcal{O}_X) - \left[\chi(\mathcal{O}_X) + \frac{(-C)(-C)}{2}\right]$$

$$= -\frac{C^2}{2}. = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - p_a(C).$$

C reduced, $\tilde{C} \xrightarrow{\nu} C$ normalization

$g(C) = g(\tilde{C})$ geometric genus

$$\Longrightarrow \quad p_a(C) = g(C) + \delta .$$

δ = length $\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$ = contribution from singularities

---

Conclusion

[i] C^2 = even

[ii] $C^2 \geq -2$.

If equality then $C^2 = -2 \Rightarrow p_a = 1 + \frac{C^2}{2} = 0$

$\Rightarrow g = 0$ & $\delta = 0$

$\Rightarrow \tilde{C} \cong \mathbb{P}^1$ & no singularities

$\Rightarrow C \cong \mathbb{P}^1.$

II. A thorougher study of the cohomology

(2) study of the lattice $H^2(X, \mathbb{Z})$

(1) cup product in $H^2(X, \mathbb{R})$

(3) Hodge decomposition of $H^2(X, \mathbb{C})$.

---

§1. Over $\mathbb{R}$

Signature

$\dim_{\mathbb{R}} X = 4k$ compact manifold

$\cup : H^{2k}(X, \mathbb{R}) \times H^{2k}(X, \mathbb{R}) \longrightarrow \mathbb{R}$ perfect pairing

type (b^+, b^-)

signature $= b^+ - b^-$.

If X is a complex surface, $k=1$

$$b^+ - b^- = \frac{1}{3}\left(K_X^2 - 2e(X)\right).$$

Aside Thom-Hirzebruch theorem

L-genus

$$b^+ - b^- = \int_X \mathcal{L}(TX)$$

This is an index theorem for $d+d^*: \Lambda^+ \longrightarrow \Lambda^-$

Compare with

$$\chi(X, \mathcal{O}_X) = \int_X td(TX)$$

This is an index theorem for the $\bar\partial + \bar\partial^*: \Lambda^{0,2k} \longrightarrow \Lambda^{0,2k+1}$

Gauss-Bonnet

$$e_{top}(X) = \int_X c(TX).$$

This is an index thm for $d+d^*: \Lambda^{ev} \longrightarrow \Lambda^{odd}$.

Total Pontryagin class $\quad V$ real vector bundle

$\boxed{i}\quad p(V) = 1 + p_1(V) + p_2(V) + \dots$

$$= \prod_i (1 + x_i)$$

$\boxed{ii}\quad p(V) = \prod_j (1 + r_j^2) \quad r_j$ chern roots of $V^{\mathbb{C}}$.

$\boxed{iii}\quad \mathcal{L}(V) = \prod_i \frac{\sqrt{x_i}}{\tanh\sqrt{x_i}} = \prod_i \left(1 + \frac{x_i}{3} - \frac{x_i^2}{45} + \dots\right)$

$$= 1 + \frac{1}{3} p_1(V) + \frac{1}{45}\left(7p_2(V) - p_1(V)^2\right) + \dots$$

$\boxed{iv}\quad p_1(TX) = \frac{1}{3}(c_1^2 - 2c_2)$ if X complex surface.

Example $X = K3$ surface $k=1$.

$$b^+ + b^- = \dim H^{2k}(X) = \dim H^2(X) = 22.$$

$$b^+ - b^- = -16$$

$$\Rightarrow \quad b^+ = 3 \quad , \quad b^- = 19$$

Exercise

If we carry out these arguments for any surface

- b_1 even $\Rightarrow b^+ = 2p_g + 1.$
- b_1 odd $\Rightarrow b^+ = 2p_g$ (not Kähler)

$p_g = \dim H^0(\Omega^2_X)$

In our case $p_g = 1.$

This shows p_g is determined by the topology.

§2. Over $\mathbb{Z}$

$\Lambda = H^2(X, \mathbb{Z})$ = lattice.

- Λ free $\mathbb{Z}$-module
- symmetric bilinear form

$\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ signature $(b^+, b^-) = (3, 19)$

<u>Remark</u> $\boxed{i}$ Λ is even $x^2 \equiv 0 \mod 2$ $\forall x \in \Lambda$.

$\boxed{ii}$ Λ unimodular $\Lambda \cong \Lambda^{\vee} = \mathrm{Hom}(\Lambda, \mathbb{Z})$.

$\Lambda \times \Lambda \longrightarrow \mathbb{Z}$. Poincaré duality.

Example [2] $\Lambda = U$ = hyperbolic lattice

$$\Lambda = \mathbb{Z}e + \mathbb{Z}f, \quad e^2 = 0, \quad f^2 = 0, \quad e.f = 1.$$

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ signature $(1,1)$

even: $Q(ae+bf) = (ae+bf)^2$

$$= a^2e^2 + b^2f^2 + 2ab\,ef$$

$$= 2ab = \text{even}.$$

$\boxed{ii}$ E_8 lattice $\rightsquigarrow$ signature $(8,0)$.

$$\Lambda = \left\{ (x_1, \dots x_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 : \sum_{i=1}^{8} x_i \equiv 0 \mod 2 \right\}$$

— vectors of length 2: — ± 1 & 2-times

— $\pm 1/2$ & 8 times

— basis

$$\begin{bmatrix} 1/2 & 1 & -1 & & & & & \\ -1/2 & 1 & 1 & -1 & & & & \\ -1/2 & & & 1 & -1 & & & \\ -1/2 & & & & 1 & -1 & & \\ -1/2 & & & & & 1 & -1 & \\ -1/2 & & & & & & 1 & -1 \\ 1/2 & & & & & & & 1 \end{bmatrix}$$

— compute the intersection matrix if you wish.

& check

- unimodular.
- even
- signature $(8,0)$

Classification of even & unimodular lattices

(Serre, A course in Arithmetic)

- sums of $\pm E_8$, U.

In our case

$$\Lambda_{K3} = (-E_8) + (-E_8) + U + U + U$$

Friedman The lattice $H^2(X, \mathbb{Z})$ determines simply connected X up to homeomorphism.

e.g. (rank, signature, parity) $\rightsquigarrow X$

§3. Over $\mathbb{C}$

Hodge decomposition, $H^{p,q} = (p,q)$-forms.

(1) $H^2(X, \mathbb{C}) = H^{2,0} + H^{1,1} + H^{0,2}$.

(2) $\overline{H^{2,0}} = H^{0,2}$.

$h^{p,q} = \dim H^{p,q}$

For us $h^{0,2} = h^{2,0} = 1$ because $h^{2,0} = \dim H^0(\Omega^2)$

$= \dim H^0(\mathcal{O})$

$\Longrightarrow h^{1,1} = 20$

$= 1.$

$H^{1,1}_{\mathbb{R}} = H^{1,1} \cap H^2(X, \mathbb{R}) \Longrightarrow H^{1,1} = H^{1,1}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$

$NS(X) = H^{1,1} \cap H^2(X, \mathbb{Z})$ Neron Severi lattice

Aside

(1) First chern class

$$Pic(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{j} H^2(X,\mathbb{C})$$

$$\mathcal{L} \longrightarrow c_1(\mathcal{L})$$

---

(2) exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 1$$ gives

$$\begin{tikzcd}
0 \arrow[d, equal] & Pic(X) \arrow[d, equal] & \\
H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1"] \arrow[d, "c_1"] & H^2(X,\mathbb{Z}) \arrow[d, "\int j"] \\
& NS \arrow[r, hook] & H^2(X,\mathbb{C})
\end{tikzcd}$$

Lefschetz thm on (1,1) classes ensures c_1 surjective.

---

(3) In our case $H^1(X,\mathcal{O}_X)=0 \Rightarrow c_1$ injective (for K3s)

$\Rightarrow \quad Pic(X) \cong NS(X)$ &

$$c_1 : Pic(X) \longrightarrow H^{1,1}_{\mathbb{R}} \text{ is injective.}$$

For K3 surfaces

We investigate the signature of the intersection form on $H^{1,1}_{\mathbb{R}}$.

Let ω be the symplectic form $H^0(X, \Omega^2_X) \cong \mathbb{C}$.

$$\underset{(2,0)}{} \quad (H^{2,0} + H^{0,2})_{\mathbb{R}} \quad \text{spanned by} \quad \omega + \bar{\omega} \ \& \ i(\omega - \bar{\omega})$$

$(\omega + \bar{\omega})^{\wedge 2} = 2\,\omega \wedge \bar{\omega} > 0$ since $\omega \wedge \omega = \bar{\omega} \wedge \bar{\omega} = 0$.

$$(i(\omega - \bar{\omega}))^{\wedge 2} = -(\omega - \bar{\omega}) \wedge (\omega - \bar{\omega}).$$

$$= 2\,\omega \wedge \bar{\omega}. > 0$$

$H^2(X, \mathbb{C}) = (H^{2,0} + H^{0,2}) + H^{1,1}$ signature $(3, 19)$

Conclusion $H^{1,1}_{\mathbb{R}}$ has signature $(1, 19)$

Consequences

Hodge Index Theorem I.

If $D^2 > 0$, $D.E = 0$ then $E^2 \leq 0$.

Proof Let V be the subspace spanned by D & E in $H^{1,1}_{\mathbb{R}}$.

If $\dim V = 2$ since $D^2 > 0 \Rightarrow V$ has signature $(1,1)$

Since $D.E = 0$ & $D^2 > 0 \Rightarrow E^2 < 0$.

If $\dim V = 1$ then $D = \mu E$ or $E = 0$. If

$D = \mu E \Rightarrow D^2 = D.D = \mu D.E = 0$. false. Thus

$E = 0 \Rightarrow E^2 = 0$.

Remark Equality happens iff $E = 0$ in $H^{1,1}_{\mathbb{R}}$

$\underbrace{\Rightarrow}_{\text{for K3s}} E \equiv 0$ since $\mathrm{Pic} \hookrightarrow H^{1,1}_{\mathbb{R}}$ injective.

Hodge Index Theorem II.

If $D_1^2 > 0$ then $D_1^2 . D_2^2 \leq (D_1 . D_2)^2$.

Proof Let

$$D = D_1^2 \cdot D_2 - (D_1 . D_2) . D_1$$

Observe that

$$D . D_1 = D_1^2 . (D_1 . D_2) - (D_1 . D_2) D_1^2 = 0 .$$

By Hodge I. $\Rightarrow$

$$D^2 \leq 0 \Rightarrow D_1^2 . D_2^2 \leq (D_1 . D_2)^2.$$

Equality occurs iff $D_2 = \mu D_1$. in $H^{1,1}_{\mathbb{R}}$ or equivalently (for a K3)

iff $D_2 \equiv \mu D_1$.

Hodge Index Theorem III. (X, H) K3 surface, H ample.

$$\mathcal{P} = \{ \alpha : \alpha^2 > 0 \} \hookrightarrow H^{1,1}_{\mathbb{R}}.$$

$\mathcal{P} = \mathcal{P}^+ \cup (-\mathcal{P}^+)$ where $\mathcal{P}^+$ is the component that contains H.

In coordinates, we can take the intersection form

$$x_0^2 - x_1^2 - \dots - x_{19}^2, \quad H = (1, 0, \dots, 0).$$

Let $\mathcal{P}^+ = \{ x_0 > 0 : x_0^2 > x_1^2 + \dots + x_{19}^2 \}.$

$\boxed{\text{L}}$ $x, y \in \mathcal{P}^+ \Rightarrow x.y > 0$

Explicitly,

$$x_0 y_0 > (x_1^2 + \dots + x_n^2)^{1/2} (y_1^2 + \dots + y_n^2)^{1/2}$$

$$\geq x_1 y_1 + \dots + x_n y_n.$$

Thus if $\mathcal{P}^+$ contains H, it contains all ample divisors

$\boxed{iii}$ $x, y \in \overline{\mathcal{P}^+} \implies x \cdot y \geq 0$

If $x \neq 0, y \neq 0$ equality occurs when $y = \lambda x$ & $x^2 = 0$.

Math 220 B - Lecture 8

February 3, 2021

Last time

Cohomology of K3s

- $\Lambda = (-E_8) + (-E_8) + U + U + U$
- $H^2(X, \mathbb{Z}) \cong \Lambda$ as lattices
- $\Lambda_{\mathbb{R}}$ has signature $(3, 19)$

Last time

Lobachevsky Geometry

$V_{\mathbb{R}}$ signature $(1,n)$

$$\mathcal{P} = \{x : x^2 > 0\} = \mathcal{P}^+ \cup (-\mathcal{P}^+).$$

$$x, y \in \mathcal{P}^+ \implies x.y > 0.$$

Thus x, y are in the same component iff $x.y > 0$.

Model $x_0^2 - x_1^2 - \ldots - x_n^2$, $\mathcal{P}^+ = \{x_0 > 0,\ x^2 > 0\}$.

The inequality $x.y > 0$ if $x^2, y^2 > 0$ is Cauchy – Schwarz.

Convention

For $V_{\mathbb{R}} = \underbrace{H^{1,1}_{\mathbb{R}}}_{(1,19)}$ or $\underbrace{NS(X)_{\mathbb{R}}}_{(1,\rho-1)}$

$\mathcal{P}^+$ = component containing one (all) ample classes.

Today

- discussion of ampleness on K3s.
- onto moduli – approach by periods

Hopefully next time

- approach by Hilbert schemes
- discussion of ampleness II ; Reider.

I. Discussion of Ample line bundles

Recall $\mathcal{L} \longrightarrow X$ ample $\iff$ $\mathcal{L}^2 > 0$ and $\mathcal{L}.C > 0$ $\forall$ C irred

(Nakai – Moishezon, H. V. 1.10).

$\mathcal{L} \longrightarrow X$ nef $\iff$ $\mathcal{L}.C \geq 0$ $\forall$ C irred $\Rightarrow$ $\mathcal{L}^2 \geq 0$

$\mathcal{L} \longrightarrow X$ big & nef $\iff$ $\mathcal{L}.C \geq 0$ and $\mathcal{L}^2 > 0$

Remark In all cases $\mathcal{L} \in \overline{\mathcal{P}^+}$

We only need to check $\mathcal{L}.C \geq 0$ for curves C, $C \notin \overline{\mathcal{P}^+}$.

so for those curves with $C^2 < 0$. We have seen in a previous lecture $C^2 \geq -2$ & C^2 even. Thus $C^2 = -2$ & C irreducible $\Rightarrow$ $C \cong \mathbb{P}^1$.

Thus in all the above inequalities we need $C \cong \mathbb{P}^1$

(For ampleness $L.C > 0$ is satisfied since equality $L.C = 0$ implies $L = \mu C$ & $C^2 = 0$ $\Rightarrow$ $L^2 = 0$ false.)

Picard - Lefschetz Reflections

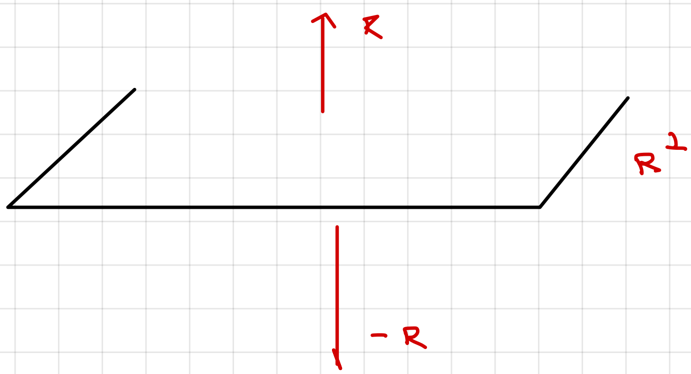
$R^2 = -2$, $\quad s_R : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}$

$$x \longmapsto x + (x.R)\,R$$

Claims (1) $s_R x . s_R y = x.y \Rightarrow s_R$ isometry

(2) $(s_R x)^2 = x^2 > 0$

(3) $s_R R = -R$, $\quad s_R\big|_{R^\perp} = \mathbb{1}$.



(4) $s_R : P^+ \longrightarrow P^+$

If $x \in P^+$, we show $s_R x \in P^+$. Note $(s_R x)^2 = x^2 > 0$

$\Rightarrow s_R x \in P$. To see $s_R x$ and x are in the same component, we only need to check

$$s_R x . x > 0 \iff x.x + (x.R)(x.R) > 0 \quad \text{which is true.}$$

Lemma Let D be a divisor with $D^2 > 0$. There are

$R_1, \dots, R_n$ such that

$\pm S_{R_1} S_{R_2} \dots S_{R_n} D$ is nef. It is furthermore ample

if $\nexists$ $R^2 = -2$ with $D.R = 0$.

Proof Note $D \in \mathcal{P}^+$ or $-D \in \mathcal{P}^+$. Let H ample

Assume WLOG $D \in \mathcal{P}^+$. Then $D.H > 0$.

If D nef, we win. Otherwise $\exists$ R_1 irreducible, $D.R_1 < 0$.

By the above remark $R_1^2 = -2$, $R_1 \cong \mathbb{P}^1$. Let $D_1 = S_{R_1} D$.

Note $D_1 . H = D.H + \underset{-}{(D.R_1)} \underset{+}{(H.R_1)} < D.H$. Also

$D_1 \in \mathcal{P}^+$ by $\boxed{IV}$ on previous page $\Rightarrow D_1 . H > 0$.

If D_1 nef, we win. If not we continue to reflect across

a new curve R_2. The process must stop since

$0 < D_1 . H < D.H$ decreases. At the end we obtain

a nef divisor.

$R . S_{R_1} \dots S_{R_n} D = 0 \iff D . \delta = 0$ for

The divisor is ample unless $\exists\ R^2 = -2$,

$\delta = s_{R_1} \cdots s_{R_n} R \Rightarrow \delta^2 = -2$. This completes the proof.

Check : the proof also works for $D^2 = 0$.

The discussion can be carried out abstractly

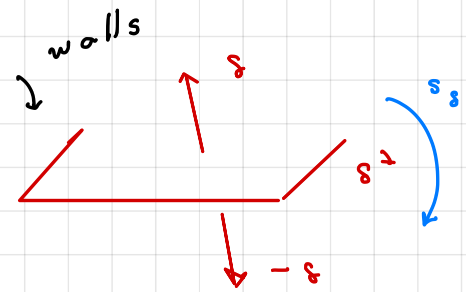
Γ lattice, $V_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$ signature $(1,n)$, $\mathcal{P}$, $\mathcal{P}^+$

e.g. $\Gamma = \Lambda_{K3}$, $NS(X)$ or other lattices (needed later)

$$\Delta = \{ \delta \in \Gamma : \delta^2 = -2 \} = \Delta^+ \cup (-\Delta^+).$$

$s_\delta : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}$, $\quad s_\delta(x) = x + (x.\delta)\,\delta$

s_δ preserves $\mathcal{P}^+$.



$$O^+(\Gamma) = \underbrace{O^+(V_{\mathbb{R}})}_{\text{preserve } \mathcal{P}^+} \cap O(\Gamma)$$

$W \leq O^+(\Gamma)$ generated by s_δ. Weyl group

Claims (1) W acts on $\mathcal{P}^+$ preserving $\bigcup_{\delta \in \Delta} \delta^\perp$

(2) W acts on $\mathcal{P}^+$ properly discontinuously

(3) $\bigcup_{\delta \in \Delta} \delta^\perp$ is closed in $\mathcal{P}^+$

$\mathcal{P}^+ \setminus \bigcup_{\delta \in \Delta} \delta^\perp$ is open. Connected components are chambers

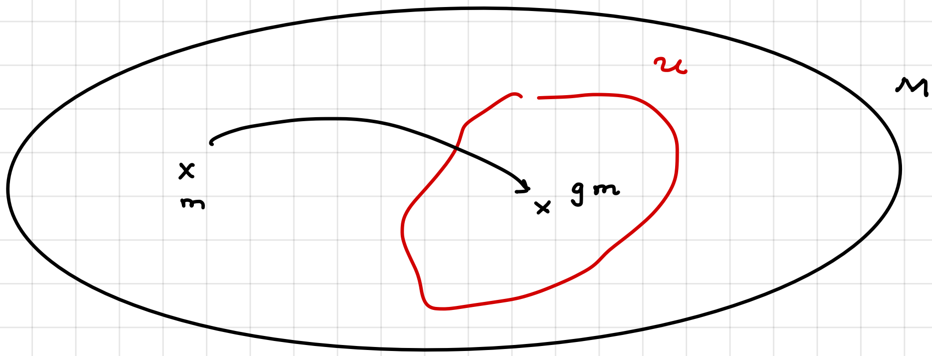
(4) W acts on chambers transitively

In addition,

(5) chamber can be taken as fundamental domain for the action of W (Vinberg, 1971).

Recall $G \curvearrowright M$ fundamental domain $\mathcal{U}$

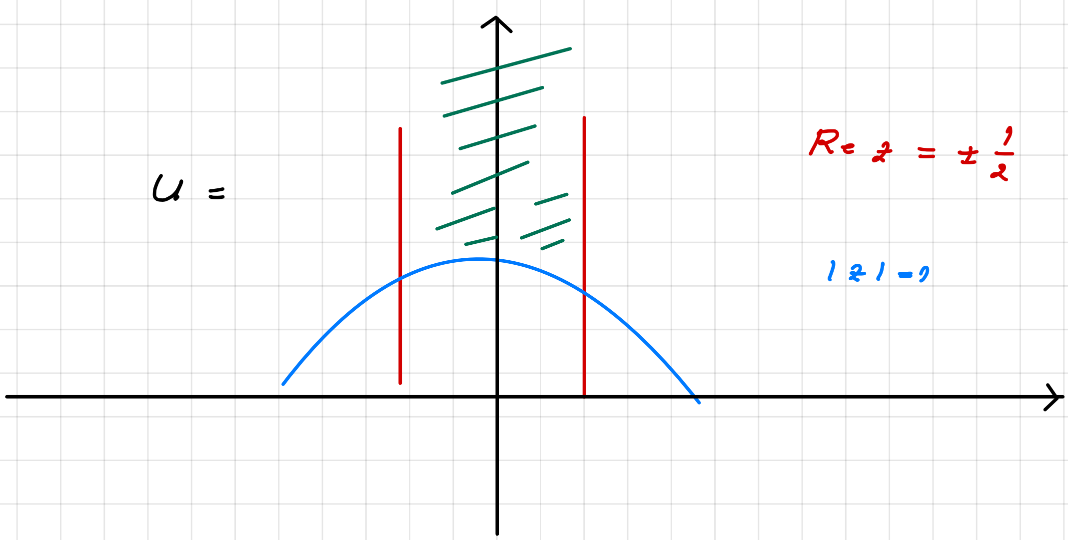
$$M = \bigcup_{g \in G} g\,\overline{\mathcal{U}} \quad \& \quad g\,\mathcal{U} \cap \mathcal{U} = \emptyset \quad \forall g \in G \setminus \{1\}.$$



For instance

$$G = PSL_2(\mathbb{Z}) \curvearrowright M = \mathfrak{H}^+$$

fundamental domain $\mathcal{U}$ bounded by

$U =$ 

Remark $\Gamma = NS(X)$ the above (5) shows

$$\mathrm{Amp}(X)_{\mathbb{R}} \hookrightarrow NS(X)_{\mathbb{R}}$$

is fundamental domain for the action of W on $\mathcal{P}_X^+$

Compare this with the Lemma.

Subtle Point A-priori in the definition of W we don't ask for δ irreducible. e.g. when $\Gamma = NS(X)$, δ may not be irred. rational curve. This is not an issue as it can be shown W is generated by $s_{[\mathbb{P}^1]}$. $\mathbb{P}^1 \hookrightarrow X$.

For instance, $\delta = R_1 + R_2$, $R_1^2 = R_2^2 = -2$, $R_1 . R_2 = 1 \Rightarrow \delta^2 = -2$.

$$\Rightarrow s_\delta = s_{R_1} s_{R_2} s_{R_1}.$$

Proof

(1) W acts on $\mathcal{P}^+$ preserving $\bigcup_{\delta \in \Delta} \delta^\perp$,

Indeed, $\forall s_\gamma \in W$: $s_\gamma : \delta^\perp \longrightarrow s_\gamma(\delta)^\perp$, $s_\gamma(\delta) \in \Delta$

Check : $s_\gamma(\delta)^2 = \delta^2 = -2$

---

(2) W acts on $\mathcal{P}^+$ properly discontinuously

$$\mathcal{P}^+ = \{ x : x^2 > 0 , x_0 > 0 \} = O^+(1,n) \big/ \underbrace{O(n)}_{\text{stabilizer of } (1,0,\dots 0)} .$$

compact.

Indeed $O^+(1,n)$ acts transitively on $\mathcal{P}^+$. & $\mathrm{Stab}_{(1,0,\dots 0)} \cong O(n)$

Fact (Topology).

$\Gamma \leq G$ discrete, $H \leq G$ (compact, locally compact) then

$\Gamma \curvearrowright G/H$ is properly discontinuous.

---

(3) $\bigcup_{\delta \in \Delta} \delta^{\perp}$ is closed in $\mathcal{P}^+$

Claim: $W \curvearrowright \mathcal{P}^+$ properly discontinuously, $S \subseteq W$ then

$$\mathcal{F} = \bigcup_{s \in S} \{x : sx = x\} \hookrightarrow \mathcal{P}^+ \text{ is closed.}$$

Take $S = \{s_\delta\}_{\delta \in \Delta}$: $\mathcal{F} = \bigcup_{\delta \in \Delta} \delta^{\perp}$ to conclude.

Proof: Let $y \in \mathcal{P}^+ \setminus \mathcal{F}$, W_y = stabilizer of y.

$\Longrightarrow$ $W_y \cap S = \phi$ by definition of $\mathcal{F}$.

Since W acts properly discontinuously,

$$\exists\; y \in \mathcal{U}, \quad g\,\mathcal{U} \cap \mathcal{U} = \phi \quad \forall g \in W \setminus W_y$$

$\hookrightarrow$ open

$\Longrightarrow$ $s\,\mathcal{U} \cap \mathcal{U} = \phi$ $\forall s \in S$ since $S \subseteq W \setminus W_y$

$\Longrightarrow$ $\mathcal{U} \subseteq \mathcal{P}^+ \setminus \mathcal{F}$. $\Rightarrow$ $\mathcal{P}^+ \setminus \mathcal{F}$ open $\Rightarrow$ $\mathcal{F}$ closed.

(4) W acts on chambers transitively

WTS $x, y \in \mathcal{P}^+ \setminus \bigcup_\delta \delta^\perp$ $\exists\, w \in W$

wx and y are in the same chamber

$\iff$ $\langle wx . \delta \rangle \langle y . \delta \rangle \geq 0 \quad \forall\, \delta^2 = -2$

Define $f : Wx \longrightarrow \mathbb{R}, \quad wx \longrightarrow \langle wx, y \rangle$.

We claim f has a minimum. at $w_0 x$. Then $\forall\, w$

$$\langle wx, y \rangle \geq \langle w_0 x, y \rangle.$$

Let $\delta^2 = -2$, $w = s_\delta w_0$. Then

$$\langle wx, y \rangle - \langle w_0 x, y \rangle \geq 0$$

$$\iff \langle s_\delta w_0 x, y \rangle - \langle w_0 x, y \rangle \geq 0$$

$$\iff \langle w_0 x, \delta \rangle \langle y, \delta \rangle \geq 0 \quad \text{as needed.}$$

Why is minimum achieved? $\forall\, x, y, a$

$K_a = \{ z : \langle z, y \rangle \leq a,\ |z| = |x| \}$ = compact. (check)

Since $W \longrightarrow P^+$; $w \longrightarrow wx$ proper (action is properly discont)

$W_* \cap K_a = \text{finite} \quad \forall a$

$\Rightarrow \{ wx : \langle wx, y \rangle \leq a \}$ finite

$\Rightarrow f$ achieves minimum

II. Onto Moduli

(1) marked K3 surfaces & periods

(2) marked polarized K3 surfaces & periods

(3) approach via Hilbert schemes.

(1) Marked K3 surfaces $\qquad \Lambda = \Lambda_{K3}$ = signature $(3,19)$. = K3 lattice

A marking of X is an isometry

$$\Phi : H^2(X, \mathbb{Z}) \longrightarrow \Lambda$$

It induces $\Phi^{\mathbb{C}} : H^2(X, \mathbb{C}) \longrightarrow \Lambda_{\mathbb{C}}$

If $\omega \in H^{2,0}(X)$ is the symplectic form, let

$$x = \Phi^{\mathbb{C}}(\omega) \in \Lambda_{\mathbb{C}},$$ well-defined up to scalars.

Period domain

$$\mathcal{D} = \{ x \in \mathbb{P}\Lambda_{\mathbb{C}} : x^2 = 0 , \ x . \bar{x} > 0 \}$$

= open in a quadric in $\mathbb{P}^{21}$

$O(\Lambda) \curvearrowright \mathcal{M} = \{ (X, \Phi) : \text{marked K3} \}$.

$$g \cdot (X, \Phi) = (X, g \circ \Phi).$$

$$\mathcal{M} = \{ (X, \Phi) \text{ marked K3s} \}$$ set

Period map $\quad Per : \mathcal{M} \longrightarrow \mathcal{D}$

$$(X, \Phi) \longrightarrow \Phi(H^{2,0}(X)).$$

We can consider Γ lattice, $\Gamma_{\mathbb{R}}$ has signature (n_+, n_-)

$$n^+ = 2 \quad \text{or} \quad n^+ = 3$$

The case $n^+ = 2$ is needed for polarized K3s.

The case $n^+ = 3$ corresponds to the K3 lattice.

Define $\mathcal{D} = \{ x \in \mathbb{P}\Gamma_{\mathbb{C}} : x^2 = 0 \,,\ x \cdot \bar{x} > 0 \}$.

Grassmannian realization

$$G^+(2, \Gamma_{\mathbb{R}}) = \{ P \subseteq \Gamma_{\mathbb{R}} \,,\ \dim P = 2,\ (\,,\,)/_P > 0 \}$$

$$G^{+,o}(2, \Gamma_{\mathbb{R}}) = \{ (P, o) : o \text{ orientation of } P \}$$

$$G^{+,o}(2, \Gamma_{\mathbb{R}}) \longrightarrow G^+(2, \Gamma_{\mathbb{R}}) \quad \text{double cover}$$

Lemma (next time)

$$\mathcal{D} \cong G^{+,0}(2, \Gamma_{\mathbb{R}}) \cong O(n_+, n_-) \big/ SO(2) \times O(n_+ - 2, n_-)$$

If $n_+ > 2$, $H = SO(2) \times O(n_+ - 2, n_-)$ is not compact.

The action of $O(\Gamma) \curvearrowright G/H$ is not expected to be properly discontinuous. When $n_+ = 2$, H = compact however.

Math 220 B — Lecture 9

February 5, 2021

Last time

- Λ = K3 lattice $= (-E_8) + (-E_8) + U + U + U$

- marked K3 surface (X, Φ)

$$\Phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$$

- Period $\mathrm{Per}(X, \Phi) = \varkappa \in \mathbb{P}\Lambda_{\mathbb{C}}$ where $H^{2,0} \hookrightarrow H^2(X, \mathbb{C})$

induces $\varkappa = \Phi^{\mathbb{C}}(H^{2,0})$ = line in $\Lambda_{\mathbb{C}}$

- Period domain

$$\mathcal{D} = \left\{ \varkappa \in \mathbb{P}\Lambda_{\mathbb{C}} : \varkappa^2 = 0 ,\ \varkappa \cdot \bar{\varkappa} > 0 \right\}.$$

- Period map

$$\mathcal{M} = \left\{ \text{marked K3 surfaces } (X, \Phi) \right\} / \sim$$

obvious notion of isomorphism

$$\mathrm{Per} : \mathcal{M} \longrightarrow \mathcal{D}$$

- $O(\Lambda)$ acts on both $\mathcal{M}$ & $\mathcal{D}$:

$$g \circ (X, \Phi) = (X, g \circ \Phi)$$

$$g \cdot \varkappa \in \mathcal{D}.$$

Period map $\quad Per : \mathcal{M} \longrightarrow \mathcal{D}.$

Fact The period map is surjective, & furthermore

$$Per : O(\Lambda) \setminus \mathcal{M} \xrightarrow{\sim} O(\Lambda) \setminus \mathcal{D}.$$

Weak Torelli Theorem

$$X \cong X' \iff \exists \ \varphi : H^2(X, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z}).$$

preserves intersection form & $\varphi_{\mathbb{C}}$ preserves Hodge decomposition

Terminology Hodge isometry

To see injectivity of Per from Weak Torelli:

$$Per\,(X, \phi) = Per\,(X', \phi') \iff \begin{tikzcd} H^2(X, \mathbb{Z}) \arrow[r, "\phi"] \arrow[d, "\varphi"] & \Lambda \\ H^2(X', \mathbb{Z}) \arrow[ur, "\phi'"'] & \end{tikzcd}$$

Let $\varphi = \Phi'^{-1} \circ \Phi$. This is a Hodge isometry. Then

$X \cong X'$. Identify $X = X'$, ϕ' & ϕ differ by an elt. in $O(\Lambda)$.

Note $O(\Lambda) \backslash \mathcal{D} \cong O(\Lambda) \backslash \mathcal{M}$ = "moduli space of K3s" (no markings).

What kind of object do we expect $O(\Lambda) \backslash \mathcal{D}$ to be?

We can consider L lattice, $L_{\mathbb{R}}$ has signature (n_+, n_-)

$n^+ = 2$ or $n^+ = 3$

Define $\mathcal{D} = \{ x \in \mathbb{P} L_{\mathbb{C}} : x^2 = 0 , x . \bar{x} > 0 \}$

Grassmannian realization

$$G^+(2, L_{\mathbb{R}}) = \{ P \subseteq L_{\mathbb{R}} , \dim P = 2, (,)/_P > 0 \}$$

$$G^{+,o}(2, L_{\mathbb{R}}) = \{ (P, o) : o \text{ orientation of } P \}$$

$$G^{+,o}(2, L_{\mathbb{R}}) \longrightarrow G^+(2, L_{\mathbb{R}}) \quad \text{double cover}$$

Claim

$$\mathcal{D} \cong G^{+,0}(2, L_{\mathbb{R}}) \cong O(n_+, n_-) \big/ SO(2) \times O(n_+ - 2, n_-)$$

Proof

$\{e, f\}$ oriented basis for P

(1) $x \in \mathcal{D}$, $x = e + if \Rightarrow P = \mathbb{R}e + \mathbb{R}f$ = oriented plane

$$x^2 = 0 \Rightarrow e^2 - f^2 = 0 \ \& \ e.f = 0$$

$$x.\bar{x} > 0 \Rightarrow e^2 + f^2 > 0 \Rightarrow e^2 = f^2 > 0 \Rightarrow \langle , \rangle|_P > 0.$$

This is well-defined after scaling $x \rightsquigarrow \lambda x$.

(2) $G^{+,0}(2, L_{\mathbb{R}})$ admits an action of $O(n_+, n_-)$

The action is transitive. Let $P = \langle v_1, v_2 \rangle$ be an oriented plane. The stabilizer of P is

$$SO(2) \times O(\langle v_1, v_2 \rangle^{\perp}) = SO(2) \times O(n_+ - 2, n_-).$$

Conclusion

$n_+ = 3$ is not expected to yield a reasonable structure on $O(L) \backslash \mathcal{D}. = O(L) \backslash \mathcal{M}$. Indeed, recall

if $\underbrace{\Gamma}_{\text{discrete}} \curvearrowright \mathcal{D} = G/H$, H compact, the action is properly discont.

In our case $G = O(n_+, n_-)$

$$H = SO(2) \times O(n_+ - 2, n_-) \quad \text{not compact}$$

$n_+ = 2$ is better behaved. H = compact

We can construct an analytic structure on $\mathcal{M}$

by gluing together deformation spaces of K3s.

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \pi \\ \mathrm{Def}(X) \end{array} \quad \text{universal deformation space. of a K3 } X.$$

It turns out $\mathrm{Def}(X)$ is smooth, 20 dim'l. $h^1(X, T_X) = 20$

However, $\mathfrak{M}$ is not Hausdorff.

Atiyah (1958)

$\exists$ $\mathcal{X}^+, \mathcal{X}^-$ $\downarrow p^+, p^-$ over Δ: families of K3 surfaces, isomorphic over $\Delta \setminus \{0\}$ but not as families over Δ.

---

(1) Fix markings of the families $\mathcal{X}^+ \xrightarrow{p_+} \Delta$, $\mathcal{X}^- \xrightarrow{p_-} \Delta$ e.g.

$$\phi^+ : R^2 p^+_* \mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}, \quad \phi^- : R^2 p^-_* \mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}$$

(2) Since $\mathcal{X}^+/_{\Delta \setminus \{0\}} \cong \mathcal{X}^-/_{\Delta \setminus \{0\}} \Rightarrow \phi^\pm : \Delta \longrightarrow \mathfrak{M}$ agree on $\Delta \setminus \{0\}$.

(3) $\mathfrak{M}$ Hausdorff $\Rightarrow \phi^+ = \phi^-$ on $\Delta \Rightarrow \mathcal{X}^+ \cong \mathcal{X}^-$ as Δ-families. In fact the two markings differ by Picard Lefschetz reflection.

Idea of the construction

$$\mathcal{X} = \{ x^2(x^2-2) + y^2(y^2-2) + z^2(z^2-2) = 2t^2 \} \hookrightarrow \mathbb{P}^3 \times \Delta$$

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \Delta \end{array}$$

family of quartics in $\mathbb{P}^3$

- $\mathcal{X}_t$ is K3 for $t \neq 0$
- $\mathcal{X}$, $\mathcal{X}_0$ singular at $p = (0,0,0)$.
- Resolve singularities! We can do so in several ways yielding different families of K3s, $\mathcal{X}^{\pm}$.

Resolve singularities

$$\begin{tikzcd} & \tilde{\mathcal{X}} \arrow[dl] \arrow[dr] & \\ \mathcal{X}^+ \arrow[dr] & & \mathcal{X}^- \arrow[dl] \\ & \mathcal{X} \arrow[d] & \\ & \Delta & \end{tikzcd}$$

$\tilde{\mathcal{X}} = Bl_p \mathcal{X}$.

The exceptional divisor $E \cong \dot{\mathbb{P}} C_{\mathcal{X},p} \cong \mathbb{P}(x^2+y^2+z^2+t^2=0) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Locally, the picture is:

$$\mathcal{X} = \{xy - zw = 0\} \subseteq \mathbb{C}^4$$

$D^+ = \{x = z = 0\} \hookrightarrow \mathcal{X}$ not Cartier

$D^+ + D^- = (x)$

$D^- = \{x = w = 0\} \hookrightarrow \mathcal{X}$ not Cartier

$\mathcal{X}^+ = Bl_{D^+} \mathcal{X}$

$\mathcal{X}^- = Bl_{D^-} \mathcal{X}$

$\tilde{\mathcal{X}} = Bl_0 \mathcal{X}$

$$\begin{tikzcd} & \tilde{\mathcal{X}} \arrow[dl] \arrow[dr] & \\ \mathcal{X}^+ \arrow[dr, "\pi^+"'] & & \mathcal{X}^- \arrow[dl, "\pi^-"] \\ & \mathcal{X} & \end{tikzcd}$$

$\mathcal{X}^+ \cong \mathcal{X}^-$ but $\mathcal{X}^+, \mathcal{X}^-$ are not isomorphic as $\mathcal{X}$-schemes.

(1) $D^{\pm}$ not Cartier on $\mathcal{X}$, but Cartier on $\mathcal{X}$.

(2) $\mathcal{X}^{\pm}$ are smooth. $\mathcal{X}^{+} = Bl_{D^{+}}\mathcal{X}$ $D^{+} = (x = z = 0)$.

Why? Equations $\mathcal{X}^{+} \hookrightarrow \mathcal{X} \times \mathbb{P}^1$ closure of Graph $\frac{x}{z} = \frac{w}{y} \cdot = \frac{X}{Z}$.

The equations of $\mathcal{X}^{+}$ in $\mathcal{X} \times \mathbb{P}^1$ are [X : Z] coordinates in $\mathbb{P}^1$.

$$xy = zw, \quad xZ = zX, \quad wZ = yX.$$

In the chart $X = 1$: $z = xZ, \; y = wZ$ so the coordinates are

(x, w, Z) ⤳ shows smoothness.

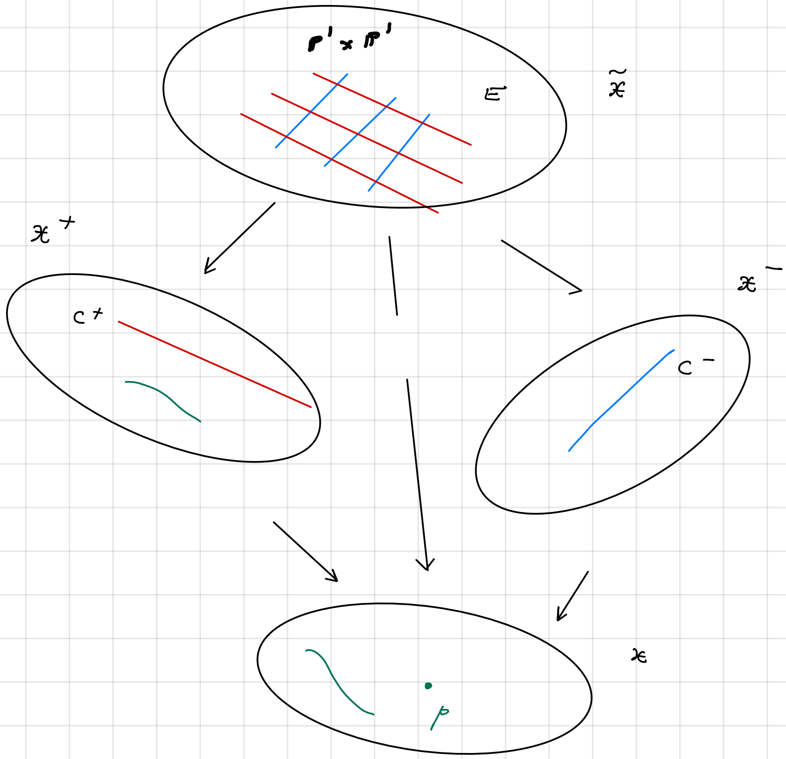
(3) $C^{\pm}$ = preimages of 0 in $\mathcal{X}^{\pm}$ are smooth $\cong \mathbb{P}^1$.

in coordinates, C^{+} corresponds to $(0, 0, Z)$.

(4) the exceptional divisor $E \cong \mathbb{P}(C_{\mathcal{X},0}) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The maps $\tilde{\mathcal{X}} \longrightarrow \mathcal{X}^{\pm}$ contract the two rulings

(5) $\mathcal{X}^{+} \cong \mathcal{X}^{-}$ but $\mathcal{X}^{+} \ncong \mathcal{X}^{-}$ as $\mathcal{X}$-varieties



$$\begin{tikzcd}
 & & E \cong \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ddll] \arrow[d, bend right] \arrow[ddrr] & & \\
 & & \tilde{\mathcal{X}} = Bl_0 \mathcal{X} \arrow[dl] \arrow[dr] & & \\
\pi^{+^{-1}}(0) = C^+ \arrow[r, hook] \arrow[ddrr] & \mathcal{X}^+ = Bl_{(x,z)} \mathcal{X} \arrow[dr, "\pi^+"] & & \mathcal{X}^- = Bl_{(x,w)} \mathcal{X} \arrow[dl, "\pi^-"'] & C^- = \pi^{-^{-1}}(0) \arrow[l, hook'] \arrow[ddll] \\
 & D^+ \arrow[r, hook] & \mathcal{X} & D^- \arrow[l, hook'] & \\
 & & 0 \arrow[u, bend left] & &
\end{tikzcd}$$

Discussion of the case $n_+ = 2$.

$$\mathcal{D} = \{ x \in \mathbb{P} L_{\mathbb{C}} : x^2 = 0 \,,\ x \cdot \bar{x} > 0 \} \,, \ L \text{ type } (2, n).$$

" Bounded symmetric domain of type $\overline{\text{IV}}$."

What does this mean? Keep $\Delta \cong \mathfrak{H}^+$ in mind.

- $D \subseteq \mathbb{C}^N$ bounded is symmetric if

$\exists\ s : D \longrightarrow D, \ s^2 = \mathbb{1}, \ \exists$ an isolated fixed point for s.

- D is irreducible if $D \ncong D_1 \times D_2$

---

4 classical & 2 exceptional (E_6, E_7)

Very rich history. Connections with

complex analysis, differential geometry, repr. thry.

...

Speak to Ming Xiao!

Harish-Chandra

Type $I_{p,q}$

$A \longrightarrow -A$ involution

$$\{ A \in Mat_{\mathbb{C}}(p,q) \; : \; I_q - A^t \cdot \bar{A} > 0 \}$$

Example $q = 1$: unit ball in $\mathbb{C}^p$.

Type II_n

$$\{ A \in Mat_{\mathbb{C}}(n,n) : I_n - A^t \cdot \bar{A} > 0 , \; A \text{ skew} \}$$

Type III_n

$$\{ A \in Mat_{\mathbb{C}}(n,n) : I_n - A^t \cdot \bar{A} > 0 : A \text{ symm} \}$$

Type IV_n — Lie Sphere

$$IV_n = \{ z \in \mathbb{C}^n : |z|^2 < \frac{1}{2} (1 + |z \cdot z|^2) < 1 \}.$$

Example $n = 1$: $\{ z \in \mathbb{C} : |z| < 1 \}. \cong \Delta.$

How does this connect with our picture

$$\mathcal{D} = \{ x \in \mathbb{P} L_{\mathbb{C}} : x^2 = 0 ,\ x \cdot \bar{x} > 0 \} \cong G^{+,0}(2, L_{\mathbb{R}})$$

$$x_0^2 + x_1^2 = z_1^2 + \dots + z_n^2 \quad \& \quad |x_0|^2 + |x_1|^2 > |z_1|^2 + \dots + |z_n|^2$$

Note $x_0/x_1 \notin \mathbb{R}$. $\Rightarrow$ 2 components $\mathcal{D}^{\pm}$

If $\operatorname{Im}(x_0/x_1) > 0$, rescale so that $x_0 - i x_1 = 1$ and write

$x_0 + i x_1 = \zeta$.

Note $|\zeta| < 1$ since ζ is the Cayley transform of $\frac{x_0}{x_1} \in \mathfrak{H}^+$.

Solving for x_0, x_1 and substituting we find

$$\Rightarrow \quad \zeta = \sum z_k^2 = z \cdot z \quad \& \quad \frac{1 + |\zeta|^2}{2} > \sum |z_k|^2$$

$$\Rightarrow \mathcal{D}^+ \cong \overline{IV}_n = \{ \sum |z_k|^2 < \tfrac{1}{2}(1 + |z \cdot z|^2) < 1 \}.$$

A different realization

$$L_{\mathbb{R}} = U_{\mathbb{R}} + W_{\mathbb{R}}, \quad U \text{ hyperbolic}$$

$$\mathcal{H} = \{ z \in W_{\mathbb{C}} : (\operatorname{Im} z)^2 > 0 \}$$

Claim $\mathcal{H} \cong \mathcal{D}$ via the isomorphism

$$z \longrightarrow [1 : -(z,z) : \sqrt{2}\, z]$$

Proof Take $x \in \mathcal{D}$. Let e, f a basis for U: $e^2 = f^2 = 0$, $e.f = 1$.

Write $x = \alpha e + \beta f + z\sqrt{2}$.

$x^2 = 0 \implies \alpha\beta + z\cdot z = 0. \quad (*)$

$x.\bar{x} > 0 \implies \alpha\bar{\beta} + \beta\bar{\alpha} + 2\, z.\bar{z} > 0. \quad (**)$

Thus $\alpha = 1$, $\beta = -(z.z)$ satisfies $(*)$. while $(**)$ gives

$$-\overline{(z.z)} - (z.z) + 2\, z.\bar{z} > 0 \iff z.\bar{z} > \operatorname{Re}(z.z) \iff \operatorname{Im} z . \operatorname{Im} z > 0$$

Example $n = 1$ $\quad \mathcal{H} = \{ \operatorname{Im} z \neq 0 \} = \mathfrak{H}^+ \cup (-\mathfrak{H}^+)$.

Marked Polarized K3s

$(X, \mathcal{L})$, $\mathcal{L}^2 = 2d$, $\mathcal{L}$ ample & primitive

---

Def Fix $\ell \in \overset{K3}{\Lambda}$, $\ell^2 = 2d$, ℓ primitive.

e.g. $\ell = e + df$, $e^2 = f^2 = 0$, $e.f = 1$.

Exercise (1) Any two such ℓ's differ by $O(\Lambda)$.

(2) $\ell^\perp = (-E_8) + (-E_8) + U + U + \underbrace{\langle -2d \rangle}_{e - df.}$

Define $\Lambda_d = \ell^\perp$. signature $(2, 19)$.

Def [I] a marked polarized K3 consists in

$$\Phi : H^2(X, \mathbb{Z}) \longrightarrow \Lambda$$

$$c_1(\mathcal{L}) \longrightarrow \ell.$$

[II] $O_d = O(\Lambda_d)$ acts on $\mathcal{M}_d = \{(X, \mathcal{L}, \Phi)\}$.

$$g \circ (X, \mathcal{L}, \Phi) = (X, \mathcal{L}, g \circ \Phi).$$

III period domain

$$\mathcal{D}_d = \{ x \in \mathbb{P}\Lambda_{\mathbb{C}} : x^2 = 0,\ x.\bar{x} > 0,\ x.\ell = 0 \}$$

$$= \{ x \in \mathbb{P}\Lambda_d^{\mathbb{C}} : x^2 = 0,\ x.\bar{x} > 0 \}.$$

IV period map is injective

$$\mathcal{M}_d \longrightarrow \mathcal{D}_d, \quad Per : O_d \backslash \mathcal{M}_d \hookrightarrow O_d \backslash \mathcal{D}_d.$$

This is a consequence of Strong Torelli

If $\exists\, \varphi : H^2(X, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z})$, $\varphi(\mathcal{L}) = \mathcal{L}'$

then $(X, \mathcal{L}) \cong (X', \mathcal{L}')$

If $Per(X, \mathcal{L}, \phi) = Per(X', \mathcal{L}', \phi')$ then strong Torelli shows we may assume $(X, \mathcal{L}) = (X', \mathcal{L}')$ and ϕ, ϕ' differ by an element in $O(\Lambda_d)$.

Theorem (Baily – Borel, Annals 1966)

$\Gamma \backslash \mathcal{D}$ is a quasiprojective variety

- also constructs compactification by adding "rational" boundary components

- the compactification is singular

- proof of projectivity uses automorphic forms

Remark The complement of the image of the period map

$$\mathcal{O}_d \setminus m_d = (\mathcal{O}_d \setminus \mathcal{D}_d) \setminus \bigcup_{\delta \in \Delta(\Lambda_d)} \delta^{\perp}$$

<u>Issue</u> : Ampleness.

(1) „$\subset$": We show $Per(X, \mathcal{L}, \phi) \notin \delta^{\perp}$. Otherwise,

$\exists\ \delta \in \Lambda_d$ with $\delta . x = 0$, $x = Per(X, \mathcal{L}, \phi) = \phi^{\mathbb{C}}(\mathcal{L})$.

$\Leftrightarrow$ $\delta . \ell = 0$ and $\delta . x = 0$. Let $R = \phi^{-1}(\delta)$. Then

$R . \mathcal{L} = 0$ and $R . \omega = 0$. $\Rightarrow R \perp H^{2,0} + H^{0,2} \Rightarrow$

$\Rightarrow R$ type $(1,1)$. Then by Lefschetz $(1,1)$ theorem,

R is a curve class. Since

$R^2 = -2 \Rightarrow \pm R$ effective. Indeed,

$$h^0 \mathcal{O}(R) + h^0 \mathcal{O}(-R) = h^0 \mathcal{O}(R) + h^2 \mathcal{O}(R) \geq \chi\, \mathcal{O}(R) = 2 + \frac{R^2}{2} = 1$$

$\Rightarrow \pm R$ is effective.

But $\mathcal{L}.R = 0$ and $\pm R$ effective contradicts $\mathcal{L}$ ample.

(2) Conversely, if $\varkappa \in \mathcal{D}_d$ then $\varkappa = Per(x, \phi)$. by the surjectivity of the period map $Per: \mathcal{M} \longrightarrow \mathcal{D}$.

Let $\mathcal{L} = \phi^{-1}(\ell)$ = integral $(1,1)$ class, primitive $\mathcal{L}^2 > 0$.

$\exists$ s_{R_1} s_{R_2} ... s_{R_n} $\mathcal{L}$ nef & in fact ample

To check ampleness we need to see that

$$\not\exists \underbrace{R}_{\text{curve}}{}^2 = -2, \quad \mathcal{L}.R = 0.$$

Let $\delta = \phi(R) \Rightarrow \ell.\delta = 0 \Rightarrow \delta \in \ell^\perp$

and $x.\delta = 0$ is automatic. since $R.\omega = 0$. Thus $x \in \delta^\perp$

for $\delta \in \ell^\perp = \Lambda_d$ which is not allowed. Therefore $\mathcal{L}$ is ample.

Math 206 — Lecture 10

February 10, 2021

§0. Last time (summary)

- $\mathcal{M}_d = \{(X, H, \Phi)\}/\sim$ = marked polarized K3s

- $\Lambda_d = \ell^\perp$, $\ell \in \Lambda_{K3}$ fixed, primitive, $\ell^2 = 2d$

- $\mathcal{D}_d = \{x \in \mathbb{P}\Lambda_d : x^2 = 0,\ x.\bar{x} > 0\}$ = period domain

- $O(\Lambda_d) \setminus \mathcal{M}_d$

  $O(\Lambda_d) \setminus \mathcal{D}_d$ & the complement is $O(\Lambda_d) \setminus \bigcup_{\substack{\delta^2 = -2 \\ \delta \in \Lambda_d}} \delta^\perp$.

- $O(\Lambda_d) \setminus \mathcal{M}_d$ = moduli of polarized K3s of degree $2d$.

  $O(\Lambda_d) \setminus \mathcal{D}_d$ = moduli of quasipolarized K3s.

Theorem (Baily – Borel, Annals 1966)

$O(\Lambda_d) \setminus \mathcal{D}_d$ is a quasiprojective variety

§1. The approach via Hilbert scheme (summary)

$(X, \mathcal{L})$, $\mathcal{L}$ ample, $\mathcal{L}^2 = 2d$.

Crucial Claim.

$\mathcal{L}^{\otimes 3}$ very ample

The proof of the crucial claim is very interesting.

Construction

$$|3\mathcal{L}| : X \longrightarrow \mathbb{P}V \quad , \quad V \cong \mathbb{C}^{9d+2}.$$

$$\dim V + 1 = h^0(3\mathcal{L}) = \chi(3\mathcal{L}) = 2 + \frac{9}{2}\mathcal{L}^2 = 2 + 9d.$$

$$P(t) = \chi\left(\mathcal{O}_{\mathbb{P}V}(t)\big/_X\right) = 2 + 9dt^2.$$

Let $Hilb = Hilb^{P}_{\mathbb{P}V}$ be the Hilbert scheme parametrizing

$$X \hookrightarrow \mathbb{P}V \quad \text{with} \quad \chi(\mathcal{O}_X(t)) = P(t).$$

Let
$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"] & Hilb \times \mathbb{P}(V) \\ Hilb & \end{tikzcd}$$
be universal family

Let
$$\begin{tikzcd} \mathcal{X}^\circ \arrow[r, hook] \arrow[d, "\pi^\circ"] & \mathcal{X} \arrow[d, "\pi"] \\ Hilb^\circ \arrow[r, hook] & Hilb \end{tikzcd}$$
be the locus where

(1) $\mathcal{X}_h$ is smooth, irreducible

(2) $h^1(\mathcal{X}_h, \mathcal{O}_{\mathcal{X}_h}) = 0$

(3) $\omega_{\mathcal{X}_h} \cong \mathcal{O}_{\mathcal{X}_h}$

(4) $p^* \mathcal{O}_{\mathbb{P}V}(1)\big|_{\mathcal{X}_h} \cong \mathcal{L}^{\otimes 3}\big|_{\mathcal{X}_h}$ for some $\mathcal{L} \in Pic_{\pi^\circ}(\mathcal{X}^\circ / Hilb^\circ)$

(5) $\mathcal{L}$ primitive

(6) $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)) \xrightarrow{\sim} H^0(\mathcal{X}_h, \mathcal{L}^{\otimes 3})$

$PGL(V) \curvearrowright Hilb^\circ$ and we need the quotient of

$Hilb^\circ$ by $PGL(V)$.

$$\mathcal{F}_d : Sch^\circ \longrightarrow Set$$

$$T \longrightarrow \left\{ \begin{array}{c} (\mathcal{X}, \mathcal{L}) \\ \downarrow \\ T \end{array} \quad \text{s.t. } (\mathcal{X}_t, \mathcal{L}_t) \text{ is primitively polarized K3 of degree } 2d \right\} \Big/ \sim$$

$$\begin{array}{c} \mathcal{X}, \mathcal{L} \\ \downarrow \\ T \end{array} \cong \begin{array}{c} \mathcal{X}', \mathcal{L}' \\ \downarrow \\ T \end{array} \iff \varphi : \mathcal{X} \xrightarrow{\sim} \mathcal{X}' \text{ over } T, \quad \varphi^* \mathcal{L}' \cong \mathcal{L} \otimes pr^* M$$

Let $\mathcal{F} : Sch^\circ \longrightarrow Set$ be such a functor.

Fine moduli space

$$\phi : \mathcal{F} \cong Hom(-, F).$$

Coarse moduli space

$$\phi : \mathcal{F} \longrightarrow h_F = Hom(-, F)$$

(1) bijective over Spec $\mathbb{C}$

(2) $\forall\, F'$ and $\psi : \mathcal{F}_d \longrightarrow h_F$

$$\begin{tikzcd} \mathcal{F}_d \arrow[r] \arrow[dr, "\phi"'] & h_F \arrow[d, dashed, "\exists !"] \\ & h_{F'} \end{tikzcd}$$

$\boxed{i}$ fine moduli space F_d

family $\begin{array}{c} (\mathcal{X}, \mathcal{I}) \\ \downarrow \\ B \end{array} \iff B \longrightarrow F_d.$

$\boxed{ii}$ coarse moduli space F_d

family $\begin{array}{c} (\mathcal{X}, \mathcal{I}) \\ \downarrow \\ B \end{array} \implies B \longrightarrow F_d$ but not conversely.

Categorical Quotient

$H = \mathrm{Hilb}^\circ$, PGL.

(1) $\cdot\ \pi : \mathcal{H} \longrightarrow F$ $\quad PGL$ - invariant

$$PGL \times \mathcal{H} \rightrightarrows H \longrightarrow F.$$

(2) $\forall\ \pi' : \mathcal{H} \longrightarrow F'$ $\quad PGL$ - invariant

$$\begin{tikzcd} H \arrow[r, "\pi"] \arrow[dr, "\pi'"'] & F \arrow[d, dashed] \\ & F' \end{tikzcd}$$

Viehweg

$\exists$ categorical quotient of Hilb° by $PGL(V)$ which is quasi projective & coarse moduli space for $\mathcal{F}_d$.

Using results of Borel, this coarse moduli space is seen to yield the same answer as the period method.

§ 2. Very ampleness on K3s

The above is a longer story. We will only prove:

Theorem $\mathcal{L} \longrightarrow X$ ample over a K3 surface. Then

[i] $\mathcal{L}^{\otimes 2}$ is globally generated

[ii] $\mathcal{L}^{\otimes 3}$ very ample

Remark A similar result is true for abelian varieties

(Think: elliptic curves are cubics in $\mathbb{P}^2$).

Question

(1) Are there generalizations for $X^{[n]}$ & $K_{n-1}(A)$?

(2) Are there generalizations for moduli of sheaves over X, A?

(3) Are there generalizations for moduli of bundles over curves?

Example

$\pi : X \longrightarrow \mathbb{P}^2$ double cover branched along sextic.

$\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}} + \mathcal{O}_{\mathbb{P}}(-3)$ Lecture

$\mathcal{L} = \pi^* \mathcal{O}(1).$

$\mathcal{L}, \mathcal{L}^{\otimes 2}$ are not very ample

Why? We have

$$
\begin{aligned}
H^0(X, \mathcal{L}^2) &= H^0(\mathbb{P}^2, \pi_* \pi^* \mathcal{O}(2)) \\
&= H^0(\mathbb{P}^2, \mathcal{O}(2) \otimes \pi_* \mathcal{O}) = \\
&= H^0(\mathbb{P}^2, \mathcal{O}(2) \otimes (\mathcal{O} + \mathcal{O}(-3))) \\
&= H^0(\mathbb{P}^2, \mathcal{O}(2)) + \underbrace{H^0(\mathbb{P}^2, \mathcal{O}(-1))}_{0} \\
&= H^0(\mathbb{P}^2, \mathcal{O}(2)).
\end{aligned}
$$

Thus $|2\mathcal{L}| : X \xrightarrow[2:1]{} \mathbb{P}^2 \xhookrightarrow[\text{Veronese}]{} \mathbb{P}^5$ is not an embedding.

Remark Matsusaka's big theorem

X smooth proj, $\mathcal{L} \longrightarrow X$ ample $\dim X = d$

$\Rightarrow \exists\, m$, $\mathcal{L}^{\otimes m}$ very ample, $m \sim L^d, L^{d-1}K$,

---

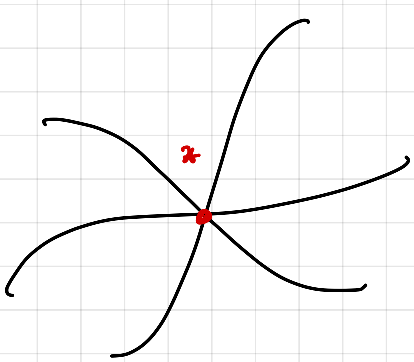
Quick reminder

$$\mathcal{L} \longrightarrow X, \quad \phi_L : X \dashrightarrow \mathbb{P}\, H^0(X, L)^\vee$$
$$x \longmapsto ev_x$$

$$Bs\,|\mathcal{L}| = \{x : \phi_L \text{ undefined}\}$$

$$= \{x : ev_x \equiv 0\}$$

$$= \{\text{all sections of } \mathcal{L} \text{ vanish at } x\}.$$



(1) $\mathcal{L}$ bpf $\iff$ $Bs\,|\mathcal{L}| = \emptyset$

(2) $\mathcal{L}$ very ample $\iff$ ϕ closed embedding.

(3) $\mathcal{L}$ ample $\iff$ $\exists\, k>0$, $\mathcal{L}^k$ is very ample.

Fujita Conjecture

$\dim X = d$, smooth $/\mathbb{C}$, $\mathcal{L}$ ample.

Then $K_X + m\mathcal{L}$ is [i] base point free if $m \geq d+1$.

[ii] very ample if $m \geq d+2$.

Remark [i] curves, $\mathcal{L}$ ample $\iff \deg \mathcal{L} > 0$ H. IV.

$\deg \mathcal{L} \geq 2g \implies \mathcal{L}$ bpf

$\deg \mathcal{L} \geq 2g+1 \implies \mathcal{L}$ very ample

$\deg K_C + m\mathcal{L} = 2g-2+m\deg\mathcal{L} \geq 2g$ if $m \geq 2 \implies$ bpf

$\deg K_C + m\mathcal{L} \geq 2g+1$ if $m \geq 3 \implies$ very ample

---

[ii] surfaces : $K_X + 3\mathcal{L}$ bpf

Reider.

$K_X + 4\mathcal{L}$ very ample.

[iii] threefolds : $K_X + 4\mathcal{L}$ bpf (Ein – Lazarsfeld).

[iv] general : $m \geq \binom{d+1}{2} + 1$. (Angehrn – Siu).

§3. Reider's Theorem

$\mathcal{L}^2 > 0$ and $\mathcal{L}$ nef. over smooth projective surface

[i] if $\mathcal{L}^2 \geq 5$ & x is a base point of $K_X + \mathcal{L}$. $\exists$ D divisor effective, $x \in D$. such that

$D.\mathcal{L} = 0$ & $D^2 = -1$. or

$D.\mathcal{L} = 1$ & $D^2 = 0$.

[ii] if $\mathcal{L}^2 \geq 10$ & $K_X + \mathcal{L}$ does not separate x & y $\exists$ D divisor effective, $x, y \in D$. such that.

$D.\mathcal{L} = 0$ & $D^2 = -2, -1$ or

$D.\mathcal{L} = 1$ & $D^2 = -1, 0$ or

$D.\mathcal{L} = 2$ & $D^2 = 0$.

Corollary A (Fujita for surfaces)

$\mathcal{L}$ ample $\Longrightarrow$ $K_X + 3\mathcal{L}$ is bpf., $K_X + 4\mathcal{L}$ very ample.

Indeed $(3\mathcal{L})^2 \geq 5$. The condition $D.\mathcal{L} \neq 0$ since $\mathcal{L}$ ample, $D \neq 0$ effective. and $D.(3\mathcal{L}) \neq 1$ for numerical reasons

$\Rightarrow K_X + 3\mathcal{L}$ bpf. The case $K_X + 4\mathcal{L}$ very ample is similar.

Corollary B (Bombieri – Kodaira)

K_X big & nef (minimal surface of general type)

$\Rightarrow$ $4K_X$ base point free.

$5K_X$ very ample away from (-2) curves.

Use $\mathcal{L} = 3K_X$ and $\mathcal{L} = 4K_X$ in Reider. We only need to rule out $K_X . D = 0$, $D^2 = -1$ which cannot happen since

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D(D-K) = \chi(\mathcal{O}_X) - \frac{1}{2} \notin \mathbb{Z}.$$

Corollary C (us + ε)

If X = K3 surface, $\mathcal{L}$ ample $\Rightarrow$ $2\mathcal{L}$ bpf & $3\mathcal{L}$ very ample.

Indeed, Reider applies since $\mathcal{L}^2 > 0$ & $\mathcal{L}^2$ even $\Rightarrow$ $(3\mathcal{L})^2 > 5$, $(4\mathcal{L})^2 > 10$.

same argument works for abelian surfaces, Enriques surfaces.

Corollary D X del Pezzo $\Rightarrow$ $-K_X$ big & nef

$|-mK_X|$ is bpf $\forall m \geq 1$. unless $m = 1$, $K_X^2 = 1$.

§4. Strategy for Reider Assume otherwise.

[i] encode the geometry in a vector bundle rank 2 over X.

[ii] study the vector bundle & show it cannot exist.

Preliminaries

$\mathcal{F}', \mathcal{F}'' \to X$. Consider extensions:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, "\wr"] & \mathcal{F}'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}$$

$\mathcal{H}om(\mathcal{F}'', \_)$:

$$\begin{array}{ccc}
\mathcal{H}om(\mathcal{F}'', \mathcal{F}'') & \xrightarrow{\delta} & \mathcal{E}xt^1(\mathcal{F}'', \mathcal{F}'). \\
\psi & & \psi \\
\mathbb{1}_{\mathcal{F}''} & \longrightarrow & e \in \mathcal{E}xt^1(\mathcal{F}'', \mathcal{F}').
\end{array}$$

Our bundle will be constructed as an extension.

§5. Warm-up – Max Noether's Theorem for curves.

Recall (H. Chp I).

$\boxed{1}$ Y normal, $Y \hookrightarrow \mathbb{P}^r$ projectively normal if

$$H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(k)) \quad \forall\, k \geq 1.$$

Y lies on the correct # of degree k hypersurfaces.

$\boxed{2}$ $\mathcal{L} \to Y$ very ample, $\mathcal{L}$ normally generated

$$\mathrm{Sym}^k H^0(Y, \mathcal{L}) \twoheadrightarrow H^0(Y, \mathcal{L}^k)$$

Recall (H. Chp IV).

$\boxed{i}$ $\deg \mathcal{L} \geq 2g+1 \implies \mathcal{L}$ very ample

$\boxed{ii}$ $C \not\cong$ hyperelliptic $\implies K_C$ is very ample.

This yields the canonical embedding

$$C \longrightarrow \mathbb{P}^{g-1}$$

Theorem

$\boxed{i}$ $\deg \mathcal{L} \geq 2g+1 \implies \mathcal{L}$ is normally generated.

$\boxed{ii}$ $C \not\cong$ hyperelliptic $\implies K_C$ is projectively normal.

find a different proof in ACGH.

Example $g = 4$: $C \hookrightarrow \mathbb{P}^3$ projectively normal

<u>Math</u> <u>206</u> — <u>Lecture</u> <u>11</u>

February 12, 2021

§1. Warm-up to Reider — Curves

C smooth projective curve, H. Chp IV.

(1) K_C is very ample

$$C \hookrightarrow \mathbb{P} H^0(K_C) \cong \mathbb{P}^{g-1}$$

(2) C hyperelliptic

$$C \xrightarrow[2:1]{} \mathbb{P}^1 \xrightarrow[\text{Veronese}]{} \mathbb{P}^{g-1}$$

Theorem (Max Noether) If C is not hyperelliptic,

(∗) is projectively normal i.e.

$$\mathrm{Sym}^k H^0(K_C) \twoheadrightarrow H^0(K_C^{\otimes k}).$$

Example $g = 4$ $\quad X \hookrightarrow \mathbb{P}^3$

$k=2$ $\quad \mathrm{Sym}^2 H^0(K_X) \twoheadrightarrow H^0(X, K_X^{\otimes 2})$

$$\begin{array}{c} \mathrm{Sym}^2 H^0(K_X) \\ \wr\wr \\ \mathrm{Sym}^2 H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \twoheadrightarrow H^0(\mathcal{O}_X(2)) \\ \wr\wr \\ H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \end{array}$$

Note $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ & $\dim H^0(X, K_X^{\otimes 2}) = 9$

$\Rightarrow \exists!$ unique quadric on which X lies.

$k = 3$: $\quad H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \twoheadrightarrow H^0(\mathcal{O}_X(3))$.

$\Rightarrow \exists!$ 5 diml space of cubics on which X lies

(4 of these are quadrics $\times$ plane) $\Rightarrow \exists$ new cubic C

Conclusion $\quad X = Q \cap C$ since degree $= 6$ on both sides.

Proof (Green – Lazarsfeld)

Easier proof is possible, but the current one generalizes further.

Suffices

$$H^0(C, K) \otimes H^0(C, K^m) \twoheadrightarrow H^0(C, K^{m+1}).$$

& use induction on m. We take $m = 1$. (hardest case)

$$H^0(C, K) \otimes H^0(C, K) \twoheadrightarrow H^0(C, K^{\otimes 2}). \text{ surjective.}$$

$$\iff \underbrace{H^0(2K)^\vee}_{\text{SII Serre duality}} \longrightarrow H^0(K)^\vee \otimes H^0(K)^\vee \quad \text{injective.}$$

$$\iff H^1(K^{-1}) \longrightarrow H^0(K)^\vee \otimes H^0(K)^\vee \quad \text{injective.}$$

$$\iff \text{Ext}^1(K, \mathcal{O}) \longrightarrow H^0(K)^\vee \otimes H^0(K)^\vee \quad \text{injective}$$

If not injective take $e \in \text{Ext}^1(K, \mathcal{O})$ in the kernel, $e \neq 0$.

Then e gives an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow K \longrightarrow 0 \qquad (1)$$

Claim This is exact on global sections

Why? $0 \longrightarrow h^0(\mathcal{O}) \longrightarrow h^0(E) \longrightarrow h^0(K) \longrightarrow h^1(\mathcal{O}) \longrightarrow \cdots$

$$h^1(\mathcal{O}) \cong h^0(K)^\vee$$

$$\Rightarrow h^0(E) = h^0(\mathcal{O}) + h^0(K) = 1 + g. \geq 3.$$

$$\det E \cong K_C$$

Fix $p \in C$. $\Rightarrow h^0(E(-p)) \geq h^0 E - 2 > 0$.

Take a section of E vanishing at $D \ni p$. This gives.

$$\mathcal{O} \xrightarrow{s} E(-D) \iff \mathcal{O}(D) \xrightarrow{s} E$$

$$0 \longrightarrow \mathcal{D} \xrightarrow{s} E \longrightarrow K\mathcal{D}^{-1} \longrightarrow 0 \quad , \quad \mathcal{D} = \mathcal{O}(D). \qquad (2).$$

by computing determinants. to identify the quotient sheaf.

Take global sections in (2).

$$(*) \quad g+1 = h^0(E) \leq h^0(D) + h^0(K-D).$$

We will show $(*)$ already proves C is hyperelliptic.

Riemann-Roch

$$1-g+\deg D = h^0(D) - h^1(D) = h^0(D) - h^0(K-D).$$

$$(*) \iff 2 + \deg D \leq 2h^0(D)$$

$$\iff 1 + \frac{\deg D}{2} \leq h^0(D).$$

Clifford's theorem (H. IV. 5.4).

special divisors

$$h^1(D) \neq 0 \implies h^0(D) \leq 1 + \frac{\deg D}{2}$$

If equality holds $\implies$ $D=0$, or $D=K$ or C hyperelliptic.

- $p \notin D = 0$ or c hyperelliptic cannot happen.

- $D = K$: $\quad 0 \longrightarrow K \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0 \qquad (2)$

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow K \longrightarrow 0 \qquad (1).$$

This would imply these extensions are split, but $e \neq 0$.

The remaining case

Riemann-Roch

If $\quad h^1(D) = 0 \implies h^0(D) = 1 - g + \deg D$

Also from (2) $\implies h^0(D) = h^0(E) \geq g+1 \implies \deg D \geq 2g.$

$$(2) \quad 0 \longrightarrow D \longrightarrow E \longrightarrow KD^{-1} \longrightarrow 0$$

$$\operatorname{Ext}^1(KD^{-1}, D) = H^1(2D - K) = H^0(2K - 2D) = 0$$

$$\implies E = D + KD^{-1} \text{ is split.}$$

Back to (1), the map $\mathcal{O} \to KD^{-1}$ is zero, for degree reasons. Thus

$$(1) \quad \begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[r, bend right, "0"'] & D + KD^{-1} \arrow[r] & \mathcal{O}_D + KD^{-1} \arrow[r] & 0. \end{tikzcd}$$

But the quotient in (1) was K not $\mathcal{O}_D + KD^{-1}$!

§ I.$\frac{1}{2}$. Aside – Proof of Clifford's Thm

$$h^1(D) \neq 0 \implies h^0(D) \leq \frac{\deg D}{2} + 1.$$

via

$$\iff h^0(D) + h^0(K-D) \leq g+1$$ Riemann–Roch

For effective divisors D, $E = K - D$

$$\Phi : |D| \times |K-D| \longrightarrow |K|$$

$$(A, B) \longrightarrow A + B$$

Φ finite because we can write $C = A + B$ in finitely many ways as effective divisors

$$\implies \left(h^0(D) - 1\right) + \left(h^0(K-D) - 1\right) \leq \underbrace{h^0(K) - 1}_{g} \implies$$

$$\implies h^0(D) + h^0(K-D) \leq g+1. \qquad (*)$$

For the equality statement, induct on $\deg D$. $\neq 0, K$.

$\deg D = 2 \implies D$ is g^1_2. $\implies C$ hyperelliptic.

Construction $\deg D \geq 4$, D gives equality : $h^0(D) = 1 + \frac{\deg D}{2} \geq 3$

Take $E \in |K - D|$, $p \in E$, $q \notin E$.

Take $D \in |D|$, $p, q \in D$.

This is possible since $h^0(D - p - q) \geq h^0(D) - 2 \geq 1$

Claim

$D' = D \cap E$ gives equality in Clifford, $\deg D' < \deg D$.

Then use induction to conclude.

Why ?

$$0 \longrightarrow \mathcal{O}(D') \longrightarrow \mathcal{O}(D) + \mathcal{O}(E) \longrightarrow \mathcal{O}(D + E - D') \longrightarrow 0$$

Then $D + E - D' = K - D'$. Take global sections:

$$h^0(D) + h^0(E) \leq h^0(D') + h^0(K - D')$$

$$\iff \underbrace{g + 1 = h^0(D) + h^0(K - D)}_{\text{equality in Clifford for } D} \leq \underbrace{h^0(D') + h^0(K - D') \leq g + 1}_{\text{showed above in } (*)}$$

$\Rightarrow$ D' gives equality in Clifford as well.

Reider

$\mathcal{L}^2 > 0$ and $\mathcal{L}$ nef. over smooth projective surface

[i] if $\mathcal{L}^2 \geq 5$ & x is a base point of $K_X + \mathcal{L}$. $\exists$ D divisor effective, $x \in D$. such that

$D.\mathcal{L} = 0$ & $D^2 = -1$. or

$D.\mathcal{L} = 1$ & $D^2 = 0$.

[ii] if $\mathcal{L}^2 \geq 10$ & $K_X + \mathcal{L}$ does not separate x & y $\exists$ D divisor effective, $x, y \in D$. such that.

$D.\mathcal{L} = 0$ & $D^2 = -2, -1$ or

$D.\mathcal{L} = 1$ & $D^2 = -1, 0$ or

$D.\mathcal{L} = 2$ & $D^2 = 0$.

Strategy for Reider

Assume otherwise.

[i] encode the geometry in a vector bundle rank 2 over X.

[ii] study the vector bundle & show it cannot exist.

§2. Sheaves on smooth surfaces (Friedman's book, Okonek's book)

(1) $\mathcal{F}$ torsion free $\Rightarrow$ $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$

---

(2) $\mathcal{F}$ reflexive if $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$

---

(3) $\mathcal{F}$ coherent $\Rightarrow$ $\mathcal{F}^{\vee}$ reflexive

---

(4) $\mathrm{Sing}(\mathcal{F}) = \{x : \mathcal{F}$ is not locally free at $x\}$.

$\mathcal{F}$ torsion free $\Rightarrow$ $\mathrm{Sing}(\mathcal{F})$ codim ≥ 2

$\mathcal{F}$ reflexive $\Rightarrow$ $\mathrm{Sing}(\mathcal{F})$ codim ≥ 3.

---

(5) X smooth surface & $\mathcal{F}$ torsion free

$\Rightarrow$ $\mathcal{F}^{\vee\vee}$ locally free and

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee} \longrightarrow \underbrace{\tau}_{\substack{\downarrow \\ \text{supported in dim } 0.}} \longrightarrow 0.$$

Example X smooth surface, $\mathcal{F}$ torsion free of rank 1.

$\mathcal{F}^{\vee\vee}$ = line bundle = $\mathcal{L}$.

$$\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee} = \mathcal{L} \implies \mathcal{F} \otimes \mathcal{L}^{-1} \hookrightarrow \mathcal{O}.$$

$$\implies \mathcal{F} \otimes \mathcal{L}^{-1} = I_Z$$

$$\implies \mathcal{F} = \mathcal{L} \otimes I_Z. \quad , Z \hookrightarrow X, \quad \dim Z = 0.$$

§ 3. Constructing vector bundles over surfaces (Serre)

(E, s) rank $E = 2$, s section, $\mathcal{L} = \det E$

$$0 \longrightarrow \mathcal{O} \xrightarrow{s} E \longrightarrow \mathcal{L} \otimes I_Z \longrightarrow 0., \quad Z \subseteq X.$$

Local picture $E/_U \cong \mathbb{C}^2 \otimes \mathcal{O}$, $s/_U = (f, g)$ regular.

$$0 \to \mathcal{O} \longrightarrow \mathcal{O} + \mathcal{O} \longrightarrow \langle f, g\rangle \longrightarrow 0.$$

$$1 \longrightarrow (f, g).$$

$$(u, v) \longrightarrow gu - fv.$$

Conversely Given $(\mathcal{L}, Z)$ can we construct a vector bundle as above?

We need $e \in \operatorname{Ext}^1(\mathcal{L} I_Z, \mathcal{O})$.

Is $\mathcal{E}$ locally free? $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{L} I_Z \to 0.$

Proposition

$\mathcal{E}$ is not locally free $\iff \exists\, Z' \subsetneq Z$,

$$e \in \operatorname{Im}\left\{ \operatorname{Ext}^1(\mathcal{L} I_{Z'}, \mathcal{O}) \longrightarrow \operatorname{Ext}^1(\mathcal{L} I_Z, \mathcal{O}) \right\}.$$

Proof

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & & \\
e \arrow[r, squiggly] & 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{L} \otimes I_Z \arrow[r] \arrow[d] & 0. & (1) \\
e' \arrow[u] \arrow[r, squiggly] & 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{E}^{\vee\vee} \arrow[r] \arrow[d] & \mathcal{L} \otimes I_{Z'} \arrow[r] \arrow[d] & 0 & (1)' \\
 & & & \tau \arrow[r, equal] \arrow[d] & \tau \arrow[d] & & \\
 & & & 0. & 0 & &
\end{tikzcd}$$

$Z' \subsetneq Z.$

The second row corresponds to an extension e' which maps to e. and $e' \in \operatorname{Ext}^1(\mathcal{L} I_{Z'}, \mathcal{O})$. Note $\mathcal{E}$ not locally free

$\iff \tau \neq 0 \iff Z' \subsetneq Z$

Corollary (Cayley – Bacharach)

If $\forall\, Z' \subsetneq Z$, $\mathrm{ext}^1(\mathcal{I}_{Z'}, \mathcal{O}) < \mathrm{ext}^1(\mathcal{I}_{Z}, \mathcal{O})$ then

$\mathcal{E}$ is locally free.

$$\Updownarrow \quad \text{Serre duality.}$$

(*) If $\forall\, Z' \subsetneq Z$: $h^1(\mathcal{I}K_X/_{Z'}) < h^1(\mathcal{I}K_X/_{Z})$.

then $\mathcal{E}$ locally free.

Take length (Z') = length $(Z) - 1$.

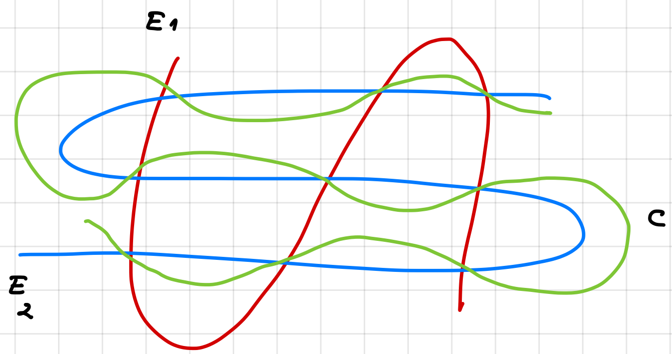
$$0 \longrightarrow I_Z \longrightarrow I_{Z'} \longrightarrow \mathbb{C}_x \longrightarrow 0$$

$$0 \longrightarrow H^0(\mathcal{I}K_X/_Z) \longrightarrow H^0(\mathcal{I}K_X/_{Z'}) \longrightarrow \mathbb{C} \longrightarrow H^1(\mathcal{I}K_X/_Z) \to H^1(\mathcal{I}K_X/_{Z'}) \longrightarrow 0$$

Condition (*) $\iff$ $H^0(\mathcal{I}K_X/_Z) \xrightarrow{\sim} H^0(\mathcal{I}K_X/_{Z'})$.

$\iff$ All sections of $\mathcal{I}K_X$ vanishing at Z' vanish at Z.

__Example__ $X = \mathbb{P}^2$; $Z = E_1 \cap E_2$ = intersection of two elliptic curves. Then any cubic C passing through 8 of the 9 intersection point passes through the last.



Math 206 — Lecture 12

February 17, 2021

Plan

- Finish Reider
- More on linear series: Hyperelliptic K3s
- Elliptic K3s

§ 0. Last time

X smooth projective surface

$\mathcal{L} \longrightarrow X$ line bundle, $Z \hookrightarrow X$ dim. zero

Serre

(1) $e \in \mathrm{Ext}^1(\mathcal{L} I_Z, \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathcal{O}_X, K_X \mathcal{L} I_Z) = H^1(K_X \mathcal{L} I_Z)$.

$$\Longrightarrow \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} I_Z \longrightarrow 0$$

(2) If $\forall\ Z' \subsetneq Z,\quad h^1(\mathcal{L} K_X I_{Z'}) < h^1(\mathcal{L} K_X I_Z)$ then

$\mathcal{E}$ is locally free (CB condition).

§ 1. Very Important Examples (will use later)

$\boxed{A}$ $Z = \phi$

If $H^1(K_X + \mathcal{L}) \neq 0$, take $e \in H^1(K_X + \mathcal{L})$, $e \neq 0$. Then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0.$$ & $\mathcal{E}$ vector bundle

This example will be used to prove

Theorem (Kawamata–Viehweg)

$\mathcal{L}^2 > 0$ nef $\Rightarrow H^1(K_X + \mathcal{L}) = 0$.

$\boxed{B.}$ $Z = \{x\}$. Assume $x \in Bs\,|K_X + \mathcal{L}|$ then

$$0 \longrightarrow K_X \mathcal{L} \otimes I_x \longrightarrow K_X \mathcal{L} \longrightarrow \mathbb{C}_x \longrightarrow 0$$ yields

$$\underbrace{H^0(K_X \otimes \mathcal{L}) \longrightarrow}_{\text{zero map}} \mathbb{C}_x \longrightarrow H^1(K_X \mathcal{L} \otimes I_x) \longrightarrow H^1(K_X \mathcal{L}) \longrightarrow 0.$$

Let $e \in H^1(K_X \mathcal{L} \otimes I_x)$, $e \neq 0$. Then we obtain

$$\Longrightarrow \quad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow K_X \mathcal{L}\, I_x \longrightarrow 0$$ & $\mathcal{E}$ vector bundle

since the Cayley - Bacharach condition is satisfied. $(Z' = \phi)$.

This example will yield Reider - 1st half.

[c] $Z = \{x, y\}$ Assume $K_X + \mathcal{L}$ does not separate x and y. Then

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}/_Z \longrightarrow 0$$ & $\mathcal{E}$ vector bdle.

This example will yield Reider - 2^{nd} half.

Step II Study the vector bundle $\mathcal{E}$

[i] stability & Bogomolov inequality.

[ii] Chern classes

§ 2. Stability of vector bundles on surfaces.

Let X be smooth projective surface

$V \to X$ torsion free, L nef divisor, $L^2 > 0$

$$\mu_L(V) = \frac{c_1 V . L}{rk\, V} = L\text{-slope}$$

Def V is L-stable if $\forall$ $0 < \text{rank}\, W < \text{rank}\, V$:

$W \hookrightarrow V$ then $\mu_L(W) < \mu_L(V)$.

Remark $rk\, W = rk\, V \Rightarrow \mu_L(W) \leq \mu_L(V)$.

Proof

Indeed, $\mathcal{W} \hookrightarrow \mathcal{V}$ gives $\mathcal{W}^{\vee\vee} \hookrightarrow \mathcal{V}^{\vee\vee}$. This is injective since the kernel $\mathcal{K}$ is torsion free, being a subsheaf of $\mathcal{W}^{\vee\vee}$ = locally free. Also $\mathcal{K}$ is supported on $\mathrm{Sing}(\mathcal{V}) \cup \mathrm{Sing}(\mathcal{W})$ which has codim 2. Thus $\mathcal{K} = 0$ and the map is injective.

Taking determinants, we find a nonzero map

$$\det \mathcal{V}^{\vee\vee} \longrightarrow \det \mathcal{W}^{\vee\vee} \iff$$

$$\iff \quad \det \mathcal{V} \longrightarrow \det \mathcal{W} \quad \text{non-zero}$$

$$\iff \quad \mathcal{O} \longrightarrow \det \mathcal{W} \cdot (\det \mathcal{V})^{\vee} \quad \text{non-zero}$$

$$\iff \quad \det \mathcal{W} \cdot (\det \mathcal{V})^{\vee} = \mathcal{O}(D), \quad D \geq 0$$

$$\implies \quad c_1(\mathcal{W}) = c_1(\mathcal{V}) + D, \quad D \geq 0.$$

$$\implies \mu_L(\mathcal{W}) = \mu_L(\mathcal{V}) + \frac{D.L}{rk} \geq 0 \quad \text{since } D.L \geq 0 \ (L \text{ nef}).$$

Lemma $\mathcal{V}$ is L-stable, $\mathcal{V} \longrightarrow Q \longrightarrow 0$ torsion free quotient with $0 < \operatorname{rk} Q < \operatorname{rk} \mathcal{V}$. $\Rightarrow \mu_L(\mathcal{V}) < \mu_L(Q)$.

Proof Let $K = \operatorname{Ker}(\mathcal{V} \longrightarrow Q)$. $\Rightarrow K$ torsion free

Write $r' = \operatorname{rk} K$, $r'' = \operatorname{rk} Q$, $\operatorname{rk} \mathcal{V} = r' + r''$

$\alpha' = c_1(K).L$, $\alpha'' = c_1(Q).L$, $\alpha = c_1(\mathcal{V}).L$.

$\Rightarrow \alpha' + \alpha'' = \alpha$.

By stability $\frac{\alpha'}{r'} < \frac{\alpha}{r}$. We need to show $\frac{\alpha}{r} < \frac{\alpha''}{r''}$.

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\frac{\alpha'}{r'} < \frac{\alpha' + \alpha''}{r' + r''} \quad \Longleftrightarrow \quad \frac{\alpha' + \alpha''}{r' + r''} < \frac{\alpha''}{r''}$$

Check.

Lemma $\Phi : V \longrightarrow V$, V = L-stable vector bundle

$$\Phi = \lambda \cdot 1\!\!1$$

Proof Let $x \in X$, $\Phi_x : V_x \longrightarrow V_x$ eigenvalue λ.

Define $\tilde{\Phi} = \Phi - \lambda \cdot 1\!\!1$. Show $\tilde{\Phi} = 0$. Assume not.

Let $Q = \operatorname{Im} \tilde{\Phi} \hookrightarrow V$ torsion free

$$\tilde{\phi} \neq 0 \implies Q \neq 0 \implies \operatorname{rk} Q \neq 0.$$

$V \twoheadrightarrow Q$ gives $\mu_L V < \mu_L Q$

$Q \hookrightarrow V$ gives $\mu_L Q < \mu_L V$

K torsion free

unless $\operatorname{rk} Q = \operatorname{rk} V \implies \operatorname{rk} K = 0 \implies K = 0 \implies \tilde{\Phi}$ is injective.

Let $\tilde{\phi} \in \operatorname{Hom}(V, V)$ have minimal polynomial

$$\tilde{\phi}^k + a_1 \tilde{\phi}^{k-1} + \dots + a_k = 0,$$

Evaluating at x & using $\tilde{\phi}$ has 0 as eigenvalue

$\implies a_k = 0.$

Using $\tilde{\Phi}$ injective

$$\Longrightarrow \tilde{\Phi}^{k-1} + \dots + a_{k-1} = 0.$$

This contradicts minimality of k. QED.

Corollary $h^0(V^\vee \otimes V) > 1 \Rightarrow V$ is not L-stable.

Remark If $\operatorname{rk} \mathcal{V} = 2$, $\mathcal{V}$ not L-stable we find

$$\mathcal{W} \hookrightarrow \mathcal{V}, \quad \operatorname{rk} \mathcal{W} = 1, \quad \mu_L(\mathcal{W}) \geq \mu_L(\mathcal{V})$$

Assume $\mathcal{V}$ is a vector bundle.

Claim We may assume $\mathcal{W}$ locally free & $\mathcal{V}/\mathcal{W}$ torsion free

---

Proof $\mathcal{W} = \mathcal{M} \otimes \mathcal{I}_Z$. $\rightsquigarrow \mathcal{W}^{\vee\vee} = \mathcal{M}$.

$\mathcal{W} \hookrightarrow \mathcal{V}$ gives $\mathcal{W}^{\vee\vee} \hookrightarrow \mathcal{V}^{\vee\vee} \cong \mathcal{V}$ & $\mathcal{W}^{\vee\vee}$ locally free

---

To achieve $\mathcal{V}/\mathcal{W}$ torsion free, we twist with a divisor $D \geq 0$.

Let $\mathcal{W}^{new} = \operatorname{Ker}\left(\mathcal{V} \longrightarrow \mathcal{V}/\mathcal{W} \longrightarrow (\mathcal{V}/\mathcal{W})/\mathrm{Tors}\right)$.

We have $\mathcal{W}^{new} = \mathcal{W}(D)$ for $D \geq 0$.

$$\begin{tikzcd}
\mathcal{W}^{new}. \arrow[dr] & & \\
\mathcal{W} \arrow[u] \arrow[r] & \mathcal{V} \arrow[r] \arrow[dr] & \mathcal{V}/\mathcal{W} \arrow[d] \\
& & (\mathcal{V}/\mathcal{W})/\mathrm{Tors}
\end{tikzcd}$$

Note $0 \to \mathcal{W}^{new} \to \mathcal{V}$ and

$$\mu_L(\mathcal{W}^{new}) = \mu_L(\mathcal{W}) + \underline{D}.L. \geq \mu_L(\mathcal{W}) \geq \mu_L(\mathcal{V}).$$

L nef

To see how the divisor D arises, work locally.

Let $Q = \mathcal{V}/\mathcal{W}^{new}$. Then $\forall x \in X$

$$\mathcal{W}_x \cong \mathcal{O}_{X,x} \longrightarrow \mathcal{V}_x = \mathcal{O}_{X,x} + \mathcal{O}_{X,x}$$

$$1 \longrightarrow (f,g).$$ Let $t = \gcd(f,g)$. in $\mathcal{O}_{X,x}$

Then Q_x has torsion supported on $t=0$. This is the germ of the divisor D at x.

Corollary $\mathcal{V}$ is not L-stable vector bundle $\Rightarrow \exists M, N$ line bundles, $A \subseteq X$ zero dimensional such that

$$0 \to M \to \mathcal{V} \to N \otimes I_A \to 0$$

where $(M-N).L \geq 0. \iff \mu_L(M) \geq \mu_L(\mathcal{V}) \geq \mu_L(N I_A).$

Bogomolov Inequality

$V \longrightarrow X$ rank r vector bundle, L-stable $\Longrightarrow$

$\Longrightarrow \quad (r-1)\, c_1^2 - 2r\, c_2 \leq 0.$

Proof for K3 surfaces

Assume $(r-1)\, c_1^2 - 2r\, c_2 > 0$. Compute

$$\chi(V^\vee \otimes V) = h^0(V^\vee \otimes V) - h^1(V^\vee \otimes V) + \underbrace{h^2(V^\vee \otimes V)}_{\text{Serre}}$$

$$= h^0(V^\vee \otimes V) - h^1(V^\vee \otimes V) + h^0(V \otimes V^\vee).$$

$$= 2h^0(V^\vee \otimes V) - h^1(V^\vee \otimes V) \leq 2h^0(V^\vee \otimes V).$$

$$\chi(V^\vee \otimes V) = \int_X ch\, V^\vee \cdot ch\, V \cdot todd(X)$$

$$= \int_X \left(r - c_1(V) + \frac{c_1(V)^2}{2} - c_2(V)\right)\left(r + c_1(V) + \frac{c_1^2(V)}{2} - c_2(V)\right)\left(1 + 2[pt]\right)$$

$$= 2r^2 + (r-1)\, c_1^2(V) - 2r\, c_2(V) \geq 4$$

$\Rightarrow h^0(V^\vee \otimes V) \geq 2.$

$\Rightarrow V$ cannot be L-stable

Sketch of proof in general (Moduli of sheaves, Thm 3.4.1).

(1) WLOG L-ample & V is L-stable

This is because we can perturb L & the strict inequalities are preserved.

---

uses complex diff. geometry

(2) V semistable $\Rightarrow$ $\operatorname{End}(V)$ semistable & $c_1(\operatorname{End} V) = 0$.

$2r c_2 - (r-1) c_1^2 = c_2(\operatorname{End} V)$ Set $W = \operatorname{End}(V)$.

---

(3) WTS: W semistable & $c_1(W) = 0 \Rightarrow c_2(W) \geq 0$. Let $s = \operatorname{rk} W$.

---

(4). $C \in |kL|$, $k \gg 0$ smooth curve. We claim:

$h^0(\operatorname{Sym}^n W(-C)) = 0$.

$h^0(\operatorname{Sym}^n W \otimes K_X(-C)) = 0$

Indeed, W semistable $\Rightarrow$ $\mathrm{Sym}^n W$ semistable (complex geometry)

If $h^0\, \mathrm{Sym}^n W(-c) \neq 0 \Rightarrow \underbrace{\mathcal{O}(c)}_{\text{positive slope}} \longrightarrow \underbrace{\mathrm{Sym}^n W}_{\text{zero slope}}$ false!

---

(5) $h^0(\mathrm{Sym}^n W) \leq h^0(\mathrm{Sym}^n W/_c) + h^0(\mathrm{Sym}^n W(-c))$

$= h^0(\mathrm{Sym}^n W/_c) + 0$ by (4)

---

(6) $h^2(\mathrm{Sym}^n W) \overset{\text{Serre}}{=} h^0(\mathrm{Sym}^n W \otimes K_X) \overset{\text{by (4)}}{\leq} h^0(\mathrm{Sym}^n W \otimes K_X/_c)$.

---

(7) Let $\mathbb{P} = \mathbb{P}(W/_c) \longrightarrow C$ be a $\mathbb{P}^{s-1}$-bundle, $s = \mathrm{rk}\, W$.

Since $\pi_* \mathcal{O}_{\mathbb{P}}(n) = \mathrm{Sym}^n W/_c$.

$\Longrightarrow$ $h^0(\mathrm{Sym}^n W/_c) = h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \leq \alpha n^s$

In general by induction on dimension, $h^0(Y, M^n) \leq \alpha \cdot n^{\dim Y}$.

---

(8) $h^0(\mathrm{Sym}^n W/_c \otimes K_X/_c) \leq \beta n^s$

---

(9) $\chi(X, \mathrm{Sym}^n W) \leq h^0(\mathrm{Sym}^n W) + h^2(\mathrm{Sym}^n W)$

$$\leq h^0(\mathrm{Sym}^n W/_C) + h^0(\mathrm{Sym}^n W/_C \otimes K_C)$$

$$\underset{(7)+(8)}{\leq} (\alpha+\beta)\, n^s$$

---

(10) Hirzebruch - Riemann - Roch

$$\chi(\mathrm{Sym}^n W) = \int_X ch\ \mathrm{Sym}^n W \cdot td(X)$$

$$\sim -\frac{n^{s+1}}{(s+1)!} \cdot c_2(W) + \ldots$$

If $c_2(W) < 0$ this grows like n^{s+1} contradicting (9).

Thus $c_2(W) \geq 0$. Q.E.D.

Math 206 — Lecture 13

February 19, 2021

§0. Last few lectures $\mathcal{L} \to X$ line bundle, nef, $\mathcal{L}^2 > 0$.

A. If $H^1(K_X + \mathcal{L}) \neq 0$ then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{L} \longrightarrow 0$$

B. If $x \in Bs\,|K_X + \mathcal{L}|$ then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{L} \otimes I_x \longrightarrow 0$$

C. If $K_X + \mathcal{L}$ doesn't separate x, y then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{L} \otimes I_{x,y} \longrightarrow 0.$$

In all cases E is rank 2 vector bundle.

If $c_1(E)^2 - 4c_2(E) > 0 \Rightarrow E$ is not L-stable

For K3s: $c_1(E)^2 - 4c_2(E) + 4 > 0 \Rightarrow E$ is not $\mathcal{L}$-stable

§ 1. Computation in examples A – C

Claim

$L^2 > 0 \implies E$ is not L-stable in A.

$L^2 \geq 5 \implies E$ is not L-stable in B.

$L^2 \geq 9 \implies E$ is not L-stable in C.

Case A $L^2 > 0$, $H^1(K_X + L) \neq 0$ yields.

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0$$

$$\left.\begin{array}{l} c_1 E = L \\ c_2 E = 0 \end{array}\right\} \implies c_1^2 - 4c_2 = L^2 > 0 \implies E \text{ is not } L\text{-stable.}$$

Case B $x \in Bs\,|K_X + L|$ yields.

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \otimes I_x \longrightarrow 0$$

$$\left.\begin{array}{l} c_1 E = L \\ c_2 E = 1. \end{array}\right\} \implies c_1^2 - 4c_2 = L^2 - 4 > 0 \implies E \text{ is not } L\text{-stable.}$$

Conclusion In examples A – C we have diagrams:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & \mathcal{O}_X \arrow[d, "s"] & & & \\
0 \arrow[r] & M \arrow[r, "\iota"'] \arrow[dr, "\varphi"'] & E \arrow[r, "\pi"'] \arrow[d, "p"] & N \otimes I_A \arrow[r] & 0 & (2) \\
 & & L \otimes I_Z \arrow[d] & & & \\
 & & 0 & & & \\
 & & (1) & & &
\end{tikzcd}$$

$$\mu_L(M) \geq \mu_L(E) \geq \mu_L(N \otimes I_A)$$

$$\Downarrow$$

(†) $M + N = L$, $(M - N) . L \geq 0$.

We can say a bit more!

Chern class calculation

$$\mathcal{M} \cdot \mathcal{N} = \text{length}(Z) - \text{length}(A).$$

Proof.

We compute $\operatorname{ch} E$ in two ways, using additivity:

$$\operatorname{ch} E = \operatorname{ch} \mathcal{M} + \operatorname{ch} \mathcal{N} \otimes I_A$$

$$= 1 + \mathcal{M} + \frac{\mathcal{M}^2}{2} + 1 + \mathcal{N} + \frac{\mathcal{N}^2}{2} - \text{length}(A)\,[\text{point}]$$

$$= \operatorname{ch} \mathcal{O} + \operatorname{ch} \mathcal{L} \otimes I_Z$$

$$= 1 + 1 + \mathcal{L} + \frac{\mathcal{L}^2}{2} - \text{length}(Z)\,[\text{point}]$$

$$\Longrightarrow \frac{\mathcal{M}^2 + \mathcal{N}^2}{2} - \text{length}(A) = \frac{\mathcal{L}^2}{2} - \text{length}(Z).$$

$$\mathcal{M} + \mathcal{N} = \mathcal{L}$$

$$\Longrightarrow \mathcal{M} \cdot \mathcal{N} = \text{length}(Z) - \text{length}(A).$$

Claim $N \geq 0$, $N \neq 0$. $\iff$ $\mathcal{L} - M \geq 0$

If $\alpha \neq 0$. $\Rightarrow$ $\alpha : M \longrightarrow \mathcal{L} I_Z \hookrightarrow \mathcal{L}$ $\Rightarrow$ $\mathcal{O} \xrightarrow{\neq 0} \mathcal{L} M^{-1}$

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\neq 0"'] & \mathcal{L} I_Z \arrow[r, hook] & \mathcal{L} \end{tikzcd}$$

$\Rightarrow$ $\mathcal{L} - M \geq 0$.

$\Rightarrow$ $N \geq 0$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & \mathcal{O}_X \arrow[d, "s"] & & & \\
0 \arrow[r] & M \arrow[r, "i"'] \arrow[ur, dashed] \arrow[dr, "\alpha"'] & E \arrow[r, "\pi"'] \arrow[d, "p"] & N \otimes I_A \arrow[r] & 0 & (2) \\
 & & \mathcal{L} \otimes I_Z \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

If $N = 0 \Rightarrow M = \mathcal{L}$. Then $\alpha : \neq 0$ gives $Z = \emptyset$

& $\alpha : \mathcal{L} \longrightarrow E$ splits the vertical extension, a contradiction.

If $\alpha = 0$ then $M \longrightarrow \mathcal{O}_X$ $\Rightarrow$ $-M$ effective

$\Rightarrow$ $-M.\mathcal{L} \geq 0$. because $\mathcal{L}$ is nef

However $(M - N).\mathcal{L} \geq 0$ $\Rightarrow$ $(2M - \mathcal{L}).\mathcal{L} \geq 0$

$\Rightarrow$ $M.\mathcal{L} \geq \frac{1}{2}\mathcal{L}^2 > 0$. contradiction!

Conclusion In cases A – C we obtain

(1) $N \geq 0, \; N \neq 0 \Rightarrow \mathcal{L}.N \geq 0$ ($\mathcal{L}$ is nef).

(2) $\mathcal{L}.(\mathcal{L} - 2N) \geq 0 \Leftarrow \mathcal{L}.(M - N) \geq 0$

(3) $N.\underbrace{(\mathcal{L} - N)}_{M} = \text{length}(Z) - \text{length}(A)$.

These will contradict Hodge Index Theorem, unless Reider etc are satisfied.

§ 2. Proof of Kawamata-Viehweg & Reider

Proof of Kawamata-Viehweg (Case A)

If $H^1(K_X + L) \neq 0$ & $L^2 > 0$, L nef. We form the diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_X & & & & \\ & & & & \downarrow s & & & & \\ 0 & \longrightarrow & L N^{-1} & \longrightarrow & E & \longrightarrow & N \otimes I_A & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & L & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

(3) $\Rightarrow$ $N(L - N) = \ell(Z) - \ell(A) = 0 - \ell(A) \leq 0.$

$\Rightarrow$ $N \cdot L \leq N^2$

(2) $\Rightarrow$ $2 N \cdot L \leq L^2$

(1) $\Rightarrow$ $N \cdot L \geq 0.$

Hodge Index Theorem

$L^2 > 0 \implies (N.L)^2 \geq N^2 . L^2$ and equality $N = \mu L$

Then $(N.L)^2 \geq N^2 . L^2 \geq (N.L)(2\, N.L) = 2\,(N.L)^2$

$\implies N.L = 0$ & $N^2 = 0 \implies N = \mu L. \implies N = 0$

contradiction.

Thus L nef & $L^2 > 0 \implies H^1(K_X + L) = 0.$

Proof of Reider Part I.

$x \in Bs\,|K_X + L|$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_X \arrow[d, "s"] & & \\
0 \arrow[r] & L N^{-1} \arrow[r] & E \arrow[r] \arrow[d] & N \otimes I_A \arrow[r] & 0 \\
 & & L \otimes I_x \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Hodge Index Theorem

$$L^2 > 0 \implies (N.L)^2 \geq N^2 . L^2.$$

(1) $N \geq 0, \ N \neq 0 \implies L . N \geq 0$

(2) $L . (L - 2N) \geq 0 \implies L^2 \geq 2 \, (N.L)$

(3) $N . (L - N) = 1 - \text{length}(A) . \leq 1.$

We may assume $N . (L - N) = 1$. because ≤ 0 was already discussed. $\implies N . L = N^2 + 1.$

Hodge: $(N.L)^2 \geq N^2 . L^2 \geq (N.L - 1) . 2 \, (N.L).$

$$\iff (N.L)^2 - 2 \, (N.L) \leq 0 \iff N.L = 0, 1, 2.$$

$$\iff N^2 = -1, 0, 1.$$

The case $N^2 = 1$, $N.L = 2$ forces $L^2 = 2$ false!

Thus $N.L = 0, N^2 = -1$ or $N.L = 1, N^2 = 0 \implies$ Reider.

Part II of Reider is very similar.

Reider's Theorem

$\mathcal{L}^2 > 0$ and $\mathcal{L}$ nef. over smooth projective surface

[i] if $\mathcal{L}^2 \geq 5$ & x is a base point of $K_X + \mathcal{L}$. $\exists$ D divisor effective, $D \neq 0$. such that

$D.\mathcal{L} = 0$ & $D^2 = -1$. or

$D.\mathcal{L} = 1$ & $D^2 = 0$.

[ii] if $\mathcal{L}^2 \geq 10$ & $K_X + \mathcal{L}$ does not separate x & y $\exists$ D divisor effective, $D \neq 0$,. such that.

$D.\mathcal{L} = 0$ & $D^2 = -2, -1$ or

$D.\mathcal{L} = 1$ & $D^2 = -1, 0$ or

$D.\mathcal{L} = 2$ & $D^2 = 0$.

Summary : We proved:

- Reider, Fujita, Kawamata - Viehweg, Kodaira - Bombieri

On K3's (abelian / Enriques / bielliptic)

$\mathcal{L}$ ample on a K3 $\Rightarrow$ $2\mathcal{L}$ bpf (only needs $\mathcal{L}$ nef, $\mathcal{L}^2 > 0$)

$\Rightarrow$ $3\mathcal{L}$ very ample

Bonus If working over K3.

$\mathcal{L}$ nef, $\mathcal{L}^2 > 0$ $\Rightarrow$ $\mathcal{L}$ bpf unless

$\exists\, D,\ D.\mathcal{L} = 1,\ D^2 = 0.$

Why? Just consider the possible cases in Reider.

The condition $\mathcal{L}^2 \geq 5$ can be improved since the proof of Bogomolov for K3s allows for a bit more slack.

Theorem: $D \neq 0$, $D^2 = 0 \Rightarrow X$ is elliptic. $X \longrightarrow \mathbb{P}^1$.

We will show this next time.

Next

(1) More on linear systems on K3s

(2) proof of theorem & a discussion of elliptic K3s.

---

§ 3. More on linear systems on K3s

Assume $\mathcal{L}^2 = 2g-2 > 0$, $\mathcal{L}$ basepoint free

$$\Phi_{\mathcal{L}} : X \longrightarrow \mathbb{P}\, H^0(X, \mathcal{L})^\vee \cong \mathbb{P}^g.$$

$$\phi_{\mathcal{L}}^* \mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{L}.$$

Remark $\mathcal{L}^2 > 0$. $\mathcal{L}$ basepointfree $\Rightarrow$ $\mathcal{L}$ nef

$h^1(X, \mathcal{L}) = 0$ by Kawamata–Viehweg.

$h^2(X, \mathcal{L}) = h^0(X, \mathcal{L}^{-1}) = 0$

$$h^0(X, \mathcal{L}) = \chi(X, \mathcal{L}) = 2 + \frac{\mathcal{L}^2}{2} = g+1.$$

Remark

$B = \operatorname{Im} \phi_{\mathcal{L}}$, $X \xrightarrow{\phi} B$, B cannot be a curve

Indeed, let $X_b = \phi^{-1}(b)$. $\Rightarrow$ $X_b^2 = 0$. Now,

$$\mathcal{L} = \sum_{b \in H \cap B} X_b \quad \Rightarrow \quad \mathcal{L}^2 = \sum_{b, b' \in H \cap B} X_b \cdot X_{b'} = 0. \quad \text{But } \mathcal{L}^2 > 0!$$

Henceforth, B is a surface (nondegenerate in $\mathbb{P}^g$).

Remark Bertini's theorem (H, Exercise $\overline{III}$. 11. 3).

generic $C \in |L|$ is smooth & irreducible.

because Φ is not composite with a pencil.

Question What can we say about this curve?

We already know genus $(C) = g$.

Lemma $\boxed{I}$ $L/_C \cong K_C$

$\boxed{II}$ $\phi_L/_C : C \longrightarrow \mathbb{P}^g$ factors through the canonical map

$$\Phi_{K_C} : C \longrightarrow \mathbb{P}^{g-1}.$$

Proof

$\boxed{i}$. Let $C \in |\mathcal{L}|$ smooth & irreducible, be given by a section $s \in H^0(X, \mathcal{L})$. We have an exact sequence

$$0 \longrightarrow T_C \longrightarrow \underbrace{T_X|_C}_{} \longrightarrow N_{C/X} = \mathcal{L}|_C \longrightarrow 0 .$$

Take determinants: $\mathcal{L}|_C \cong K_C$.

$\boxed{ii}$

$$0 \longrightarrow \mathcal{O} \xrightarrow{\cdot s} \mathcal{L} \longrightarrow \mathcal{L}|_C \longrightarrow 0 .$$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\cdot s} H^0(X, \mathcal{L}) \longrightarrow H^0(\mathcal{L}|_C) \longrightarrow H^1(\mathcal{O}) = 0 .$$

$$H^0(\mathcal{L}|_C) \overset{\boxed{i}}{=} H^0(K_C)$$

This proves $\phi_{\mathcal{L}}|_C : C \longrightarrow \mathbb{P}\, H^0(K_C)^\vee$ is the canonical embedding.

Question What can we say about Φ_2 and its image?

Recall (del Pezzo - Math 203 A - G & H)

If S' is a reduced irred nondegenerate variety in $\mathbb{P}^g$ then

$$\deg S \geq \operatorname{codim} S + 1. \quad \text{(induct on dimension)}$$

For surf**aces**

Equality only occurs if ($S = \mathbb{P}^2 \hookrightarrow \mathbb{P}^2$ or)

$\boxed{i}$ Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, $\deg = 4$, $\operatorname{codim} = 3$

$\boxed{ii}$ rational normal scrolls

---

del Pezzo's Thm

Conclusion $\underbrace{B = \operatorname{Im}\phi_L}_{\text{surface}} \hookrightarrow \mathbb{P}^g \Rightarrow \deg(B \to \mathbb{P}^g) \geq g-1.$

$$2g-2 = \deg(X \longrightarrow \mathbb{P}^g) = \deg(X \longrightarrow B)\ \underbrace{\deg(B \longrightarrow \mathbb{P}^g)}_{\geq g-1}.$$

$$\Rightarrow \deg(X \xrightarrow{\phi_L} B) \leq 2.$$

If $\deg(X \longrightarrow B) = 2$ then B must be $\mathbb{P}^2$ or a rational normal scroll.

Otherwise $\deg(X \longrightarrow B) = 1$.

To be continued next time.

Math 206 — Lecture 14

February 24, 2021

§0. Last time

X = K3 surface, $\mathcal{L} \to X$, $\mathcal{L}^2 > 0$, base point free

$\Phi_{\mathcal{L}} : X \longrightarrow \mathbb{P}^g$, $\mathcal{L}^2 = 2g-2. > 0$

$\Longrightarrow$ $\deg \Phi_{\mathcal{L}} = 1$ or 2 & if degree is 2 then

$B = \Phi_{\mathcal{L}}(X) \Longrightarrow \deg\left(B \hookrightarrow \mathbb{P}^g\right) = g-1.$

If S' is a reduced irred nondegenerate variety in $\mathbb{P}^g$ then

$\deg S \geq \operatorname{codim} S + 1.$

Further discussion

$\mathcal{L} \longrightarrow X$ basepoint free, $\mathcal{L}^2 > 0$

$C \in |\mathcal{L}|$ smooth & irreducible

$\Phi_{\mathcal{L}}\big|_C : C \longrightarrow \mathbb{P}^{g-1}$ is the canonical map

---

H. IV.

Recall : if C is hyperelliptic then

$\Phi_{K_C} : C \longrightarrow \mathbb{P}^{g-1}$ has degree 2 onto its image,

else it is an isomorphism onto its image.

(1) $\deg \Phi_2 = 2$. & generic $C \in |L|$ is hyperelliptic

$\Rightarrow$ (X, L) is said to be hyperelliptic.

(2) else $\deg \Phi_2 = 1$. & generic $C \in |L|$ is not hyperelliptic

Φ_2 may contract curves D, $L \cdot D = 0 \Rightarrow D^2 = -2$

& the image B may be singular.

§1. The nonhyperelliptic case

The following result bears analogies with Max Noether's

Theorem $(X, \mathcal{L})$ is not hyperelliptic

$\Rightarrow$ $\mathcal{L}$ is normally generated

$$\mathrm{Sym}^k H^0(X, \mathcal{L}) \twoheadrightarrow H^0(X, \mathcal{L}^k) \text{ surjective.}$$

Proof Take $C \in |\mathcal{L}|$ smooth & irred. & not hyperelliptic.

$s \in H^0(\mathcal{L})$ that cuts out C.

last time.

$$(1) \quad 0 \longrightarrow \mathcal{O} \xrightarrow{\cdot s} \mathcal{L} \longrightarrow \mathcal{L}/_C \longrightarrow 0 \quad , \quad \mathcal{L}/_C \cong K_C$$

$$\Rightarrow (2) \quad 0 \longrightarrow \mathcal{L}^k \longrightarrow \mathcal{L}^{k+1} \longrightarrow \underbrace{\mathcal{L}^{k+1}/_C}_{\substack{\cong \\ K_C^{\otimes k+1}}} \longrightarrow 0 .$$

We have $H^1(\mathcal{L}^k) = 0$ by Kawamata–Viehweg. Thus taking cohomology, we obtain from (2):

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{L}^k) \arrow[r, "\cdot s"] & H^0(\mathcal{L}^{k+1}) \arrow[r] & H^0(K_C^{k+1}) \arrow[r] & 0 \\
& \mathrm{Sym}^k H^0(L) \arrow[r, "\cdot s"] \arrow[u, two heads] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Sym}^{k+1} H^0(L) \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Sym}^{k+1} H^0(K_C) \arrow[r] \arrow[u, two heads] & 0.
\end{tikzcd}$$

surjective by induction

surjective by Max Noether.

surjective since

$H^0(L) \longrightarrow H^0(K_C) \longrightarrow H^1(\mathcal{O}) = 0.$

from (1).

The middle map is surjective by diagram chase.

§2. The Hyperelliptic case

$\boxed{\text{I}}$ $X \xrightarrow[2:1]{\pi} \mathbb{P}^1 \times \mathbb{P}^1$ branched along smooth $(4,4)$ curve

$\mathcal{L} = \pi^*(h_1 + h_2)$

$E = \pi^* h_1$

$\Rightarrow$ $\mathcal{L}.E = 2.$

$\mathcal{L}^2 = 4$

$E^2 = \pi^* h_1^2 = 0$

$\boxed{\text{II}}$ $X \xrightarrow[2:1]{\pi} \mathbb{F}_1$ branched along a smooth

$\Sigma \in |-2K_{\mathbb{F}_1}| = |4\sigma_\infty + 6f|$

$\mathcal{L} = \pi^*(\sigma_\infty + kf)$

$E = \pi^* f$

$\Rightarrow$ $\mathcal{L}^2 = 2(-1 + 2k) = 4k-2.$

$\mathcal{L}.E = 2.$

$E^2 = \pi^* f^2 = 0.$

Saint - Donat

$\mathcal{L}^2 \geq 4$, $\mathcal{L}$ base pointfree & primitive. Then

$(X, \mathcal{L})$ hyperelliptic

$\Longleftrightarrow \quad \exists \; E$ with $E.\mathcal{L} = 2$, $E^2 = 0$.

If B is a reduced irred nondegenerate variety in $\mathbb{P}^g$ then

$$\deg B \geq \operatorname{codim} B + 1.$$

For surfaces (Griffiths & Harris, Chp 5).

Equality only occurs if ($B = \mathbb{P}^2 \hookrightarrow \mathbb{P}^2$ or)

[i] Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. $\deg = 4$, $\operatorname{codim} = 3$

[ii] rational normal scrolls

Rational Normal Scrolls

$$B = \mathbb{F}_n \qquad \sigma_\infty^2 = -n, \quad \sigma_\infty \cdot f = 1, \quad f^2 = 0$$

$$\begin{tikzcd} \mathbb{F}_n \arrow[d] \\ \mathbb{P}^1 \arrow[u, bend right, "\sigma_\infty"'] \end{tikzcd}$$

$L = \sigma_\infty + (n+r) f$ is bpf if $r \geq 0$ (Lecture 6).

very ample if $r > 0$.

$\chi(L) \overset{HRR}{=} n + 2r + 2$. & $h^1(L) = h^2(L) = 0$ using the natural exact sequences.

$|L| : \mathbb{F}_n \longrightarrow \mathbb{P}^{n+2r+1}$ degree $L^2 = (\sigma + (n+r)f)^2 = n + 2r$.

embedding for $r > 0$. codimension $n + 2r - 1$

Let $\mathbb{F}_{n,r}$ denote the image $\cong \mathbb{F}_n$.

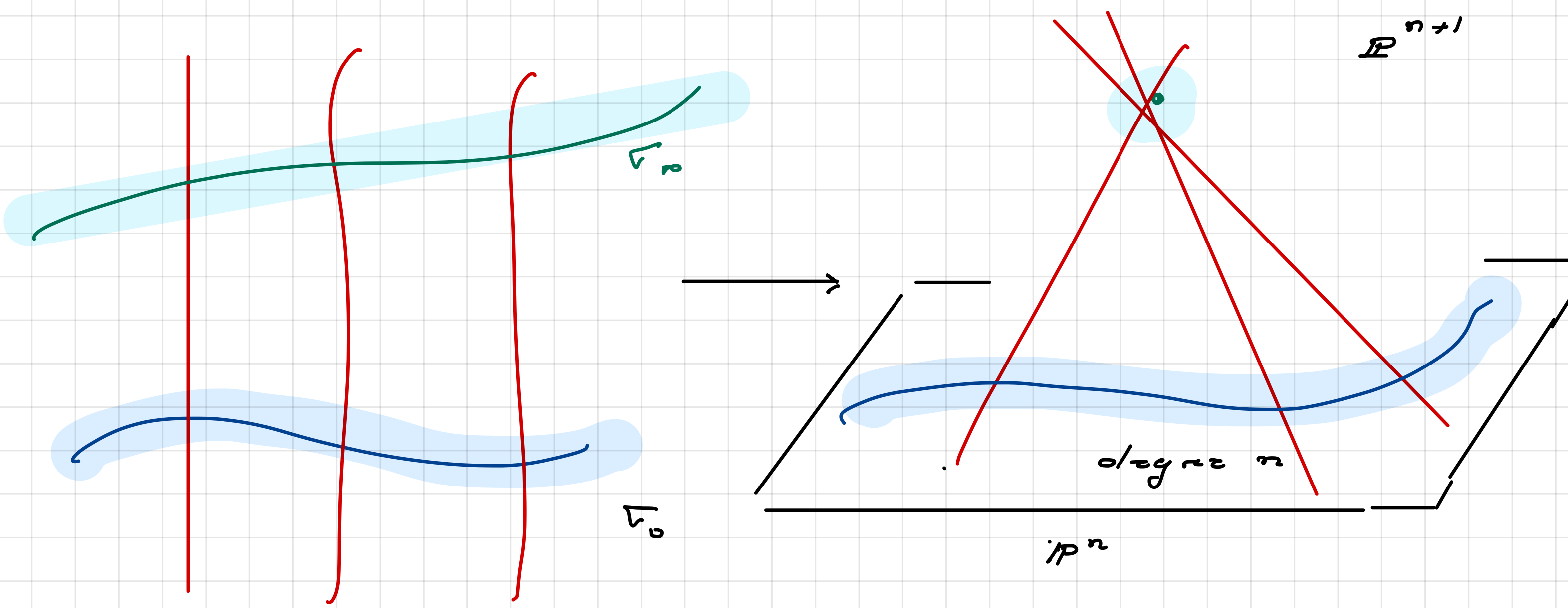
We obtain $\mathbb{F}_{n,r} \hookrightarrow \mathbb{P}^{n+2r+1}$, $L = \sigma_\infty + (n+r) f$.

$$\Rightarrow \deg \mathbb{F}_{n,r} = \operatorname{codim} \mathbb{F}_{n,r} + 1.$$

When $r=0$, $\mathbb{F}_n \longrightarrow \mathbb{P}^{n+1}$ & let $\mathbb{F}_{n,0}$ be the image. $\mathbb{F}_{n,0} \longrightarrow \mathbb{P}^{n+1}$ has degree n, codim $n-1$

$$\Rightarrow \deg = \operatorname{codim} + 1.$$

Note $L \cdot \Gamma_\infty = 0 \Rightarrow \mathcal{L}$ contracts Γ_∞



$\Gamma_0 = \Gamma_\infty + nf \in |\mathcal{L}|$. $\Rightarrow \Gamma_0$ is a hyperplane section

$\mathbb{F}_{n,0}$ is a cone over Γ_0 = rational normal curve in $\mathbb{P}^n$.

If $(X, \mathcal{L})$ hyperelliptic then $\Phi_{\mathcal{L}} : X \longrightarrow B$ where

$B = \mathbb{P}^2$ or B = rational normal scroll. $\mathbb{F}_{n,r}$ with

---

(1) $X \longrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \qquad \mathcal{L}^2 = 2$

(2) $X \longrightarrow \mathbb{P}^2 \xhookrightarrow{\mathcal{O}(2)} \mathbb{P}^5 \qquad \mathcal{L}$ not primitive.

(3) $X \xrightarrow[\pi]{2:1} \mathbb{F}_{n,r} \cong \mathbb{F}_n \hookrightarrow \mathbb{P}^{n+2r+1} \quad (r > 0).$

$$\mathcal{L} = \pi^*(\sigma_\infty + (n+r) f)$$

$$E = \pi^* f. \implies E^2 = \pi^* f^2 = 0$$

$$\mathcal{L} . E = 2$$

When $r = 0$, work with $\mathcal{L} + \pi^* f$ and map to $\mathbb{F}_{n,1}$

and find E this way.

Dolgachev - Reid : showed $0 \leq n \leq 4$.

Outcome $\mathcal{L}^2 > 0$, $\mathcal{L}$ nef

(1) $\mathcal{L}$ not bpf $\Rightarrow$ $\exists$ $D^2 = 0$, $D.\mathcal{L} = 1$.

last time.

(2) $\mathcal{L}$ bpf & hyperelliptic $\Rightarrow$ $\exists$ $D^2 = 0$, $D.\mathcal{L} = 2$

($\mathcal{L}$ primitive, $\mathcal{L}^2 \geq 4$)

(3) else $\Phi_{\mathcal{L}}$ is birational & may contract (-2) curves

§ 3. The elliptic case

Question What does the condition $D^2 = 0$ mean?

Theorem A $D \neq 0$, $D^2 = 0 \Rightarrow X$ is an elliptic fibration $X \longrightarrow \mathbb{P}^1$.

Theorem A' $D \neq 0$, $D^2 = 0$, D nef $\Rightarrow$

$\Rightarrow X$ elliptic fibration $X \longrightarrow \mathbb{P}^1$, $D = mf$.

Theorem A'' $\mathcal{L}^2 > 2$ nef, $\mathcal{L} . D = 1$, $D^2 = 0$ $\Rightarrow$

$\Rightarrow X$ elliptic fibration $X \longrightarrow \mathbb{P}^1$ with section σ

$\mathcal{L} = \sigma + mf$

Remark

It can be shown that if $p \geq 5 \Rightarrow X$ elliptic.

We first prove Theorem A' & Theorem A.

Example $X \longrightarrow \mathbb{P}^1$, elliptic fibration, $\mathcal{L} = mf$,

Claims $\boxed{i}$ $h^0(\mathcal{L}) = m+1 \Rightarrow h^1(\mathcal{L}) = m-1$. since $\chi(\mathcal{L}) = 2$.

$\boxed{ii}$ $\mathcal{L}$ base point free $\Rightarrow \mathcal{L}$ nef

Item $\boxed{i}$ follows by induction on m using

$$0 \longrightarrow \mathcal{O}((m-1)f) \longrightarrow \mathcal{O}(mf) \longrightarrow \mathcal{O}(mf)/_f \cong \mathcal{O}_f \longrightarrow 0$$

$\Rightarrow h^0(\mathcal{O}(mf)) \leq h^0\big(\mathcal{O}((m-1)f)\big) + h^0(\mathcal{O}_f) \leq m+1$. Note that

if $x_1, \dots x_m \in \mathbb{P}^1$ then $f_{x_1} + \dots + f_{x_m}$ is a section of $\mathcal{O}(mf)$

$\Rightarrow h^0\, \mathcal{O}(mf) \geq 1+m$.

Then $\mathbb{P} H^0(\mathcal{O}(mf)) = \mathbb{P}^m \cong \mathrm{Sym}^m \mathbb{P}^1$ corresponding to the choice of m points $x_1, \dots x_m \in \mathbb{P}^1$ & fibers over them.

This shows $Bs\,|mf| = \emptyset$ and

$$|mf| : X \longrightarrow \mathbb{P}^m \text{ factors } X \longrightarrow \mathbb{P}^1 \underbrace{\longrightarrow}_{\text{Veronese}} \mathbb{P}^m.$$

Example

$$X \underset{\sigma}{\overset{f}{\rightleftarrows}} \mathbb{P}^1 \qquad \mathcal{L} = \sigma + mf, \quad m \geq 2.$$

Claims [i] $\mathcal{L}$ nef, $\mathcal{L}^2 > 0$

[ii] $\mathcal{L}$ is not base point free.

If $\mathcal{L}.\underset{\text{irred.}}{C} < 0 \Rightarrow (\sigma + mf).C < 0 \Rightarrow \sigma.C < 0$ or $\sigma.f < 0$.

$\Rightarrow C \subseteq \sigma$ or $C \subseteq f$. $\Rightarrow C = \sigma$ or $C \subseteq f$. But

if $C = \sigma$, $\mathcal{L}.\sigma = m - 2 \geq 0$. If $C \subseteq f \Rightarrow C.f = 0$ by.

picking a fiber $\tilde{f}$ avoiding C so that $C.f = C.\tilde{f} = 0$.

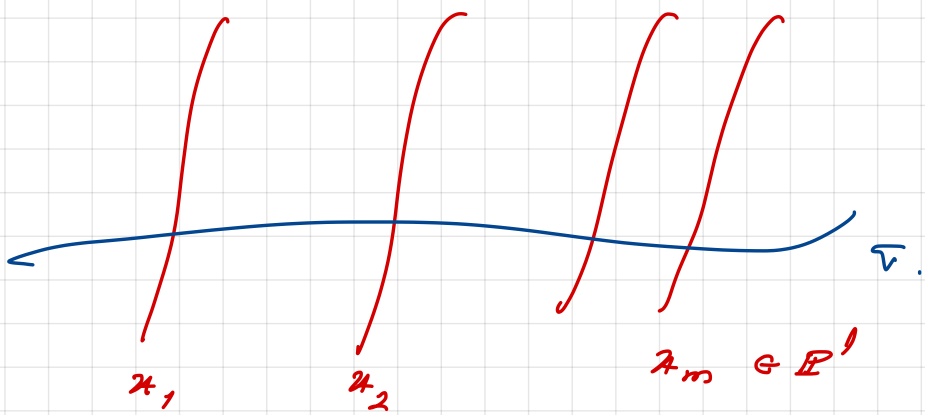
thus $\mathcal{L}$ nef & $\mathcal{L}^2 = 2m-2$. Thus

$$h^1(\mathcal{L}) = h^2(\mathcal{L}) = 0.$$

Since $\chi(\mathcal{L}) = 2 + \frac{\mathcal{L}^2}{2} = m+1 \Rightarrow h^0(\mathcal{L}) = m+1 = h^0(mf)$.

The divisors in $|\mathcal{L}|$ are $\tau + f_{x_1} + \dots + f_{x_m} \Rightarrow$

$\Rightarrow Bs\,|\mathcal{L}| = \tau$.



Math 220B — Lecture 15

February 26, 2021

§0. Goal Today –

The theorem of Piatetski – Shapiro & Shafarevich.

Theorem A $D \neq 0$, $D^2 = 0$ $\Rightarrow$ X is an elliptic fibration $X \longrightarrow \mathbb{P}^1$.

Theorem A' $D \neq 0$, $D^2 = 0$, D nef $\Rightarrow$

$\Rightarrow$ X elliptic fibration $X \longrightarrow \mathbb{P}^1$, $D = mf$.

---

§ 1. Theorem A' $\Rightarrow$ Theorem A

Recall from Lecture 8,

Lemma Let D be a divisor with $D^2 \geq 0$. There are $R_1, \ldots, R_n$ such that

$$D' = \pm S_{R_1} S_{R_2} \cdots S_{R_n} D \text{ is nef.}$$

where S_R are reflections $S_R(D) = D + (D.R)R$.

Note $S_R(D)^2 = D^2 \Rightarrow D'^2 = D^2 = 0$, D' nef.

By Theorem A', $X \longrightarrow \mathbb{P}^1$ elliptic, $D' = mf$

We furthermore see that

$$D = \pm S_{R_n} \cdots S_{R_1} D' = \pm \left(mf + \sum \alpha_i R_i\right).$$

for some m and some α_i.

§2. Proof of Theorem A' — Preliminaries

$\mathcal{L}^2 = 0, \ \mathcal{L} \neq 0 \implies \exists X \longrightarrow \mathbb{P}^1$ elliptic fibrations.

Terminology

Let $|\mathcal{L}| \neq \phi$.

$Bs\,|\mathcal{L}|$ could have components of dim 0 or 1.

Let F be the union of all 1-dim'l components

$\implies F =$ fixed part of $|L|$.

$M = L - F =$ mobile part. Note $h^0(L) = h^0 L(-F)$ since all sections of L vanish at F. This shows $h^0(L) \cong h^0(M)$ and $Bs\,|M| =$ zero dimensional components.

Example $X \underset{f}{\rightleftarrows} \mathbb{P}^1$ elliptic with section.

$$\sigma^2 = -2, \quad \sigma . f = 1, \quad f^2 = 0$$

Let $\mathcal{L} = \sigma + mf, \quad m \geq 2$

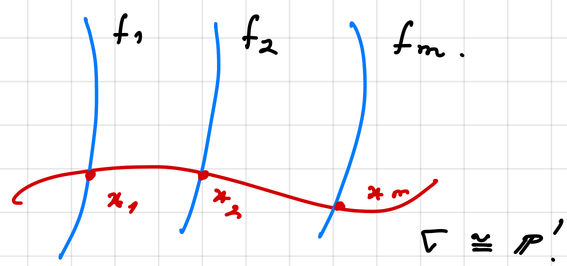
$$\mathcal{L}.\sigma = m-2, \quad \mathcal{L}.f = 1, \quad \mathcal{L}^2 = 2m-2 > 0$$

$\mathcal{L}$ big & nef $\Longrightarrow h^1(\mathcal{L}) = h^2(\mathcal{L}) = 0$.

$\chi(\mathcal{L}) = 2 + \frac{\mathcal{L}^2}{2} = m+1 \Rightarrow h^0(\mathcal{L}) = m+1$.

$\mathbb{P}^m \cong |\mathcal{L}|$ consists in

$$\sigma + f_1 + \dots + f_m.$$



Here σ is the fixed part, mf is the mobile part

Lemma [i] $h^0(L) = h^0(L-F) = h^0(M) \neq 0$.

[ii] M bpf $\Rightarrow$ M mobile $\Rightarrow$ M nef

[iii] $h^0(F) = 1$

Proof $\boxed{ii}$ M bpf $\Rightarrow Bs\, M = \phi \Rightarrow M$ mobile

M mobile $\Rightarrow M$ nef. If $M.C < 0$, C irreducible,

let $C' \in |M|$. $\Rightarrow C.C' < 0 \Rightarrow C$ is component of C' $\forall C' \in |M|$

$\Rightarrow C \subseteq Bs\,|M|, = 0\text{-dim}l.$, contradiction.

Proof $\boxed{iii}$ Assume $h^0(\mathcal{O}(F)) \neq 1. \Rightarrow h^0\, \mathcal{O}(F) \geq 2.$

Let $F' \equiv F$, $F' \neq F$, F' effective. Then by $\boxed{i}$,

$$h^0(L) = h^0\, L(-F) = h^0\, L(-F').$$

Thus $0 \to H^0(L(-F')) \longrightarrow H^0(L)$. is an isomorphism, so

all sections of L vanish at $F' \Rightarrow F' \subseteq Bs\,|L| \Rightarrow F' \subseteq F.$

But $F - F' \equiv 0$ & $F - F' \geq 0 \Rightarrow F - F' = 0 . \Rightarrow F = F'!$

Contradiction! Then $h^0\, \mathcal{O}(F) = 1.$

Lemma (should have proven a while back)

$$\mathcal{L}^2 = 0, \ \mathcal{L} \neq 0, \ \mathcal{L} \text{ nef} \Rightarrow h^0(\mathcal{L}) \geq 2 \Rightarrow \mathcal{L} \text{ effective.}$$

Proof Serre duality and Riemann-Roch:

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^{-1}) = h^0(\mathcal{L}) + h^2(\mathcal{L})$$

$$= \chi(\mathcal{L}) = 2 + \frac{\mathcal{L}^2}{2} = 2.$$

If $h^0(\mathcal{L}^{-1}) \neq 0 \Rightarrow \mathcal{L}^{-1} = \mathcal{O}(C), \ C \geq 0, \ C \neq 0$

$\Rightarrow \mathcal{L}.H = -C.H < 0$. But $\mathcal{L}$ nef $\Rightarrow \mathcal{L}.H \geq 0$

since $|mH|$ contains curves. Contradiction!

Thus $h^0(\mathcal{L}^{-1}) = 0 \Rightarrow h^0(\mathcal{L}) \geq 2$.

§ 3. Proof of theorem A'

Lemma.

$\mathcal{L}^2 = 0$, $\mathcal{L} \neq 0$, $\mathcal{L}$ nef $\Rightarrow$ $\mathcal{L}$ effective

WTS: $\mathcal{L} = \mathcal{O}(mf)$ for $X \longrightarrow \mathbb{P}^1$. $h^0(L) \geq 2$.

Steps

$\boxed{\text{I}}$ $\mathcal{L}$ mobile $B = |\mathcal{L}| = 0$-diml.

$\boxed{\text{II}}$ $\mathcal{L}$ basepoint free

$\boxed{\text{III}}$ $\phi_{\mathcal{L}} : X \longrightarrow \mathbb{P}^1$.

Step $\boxed{\text{I}}$ $\mathcal{L} = M + F$, M mobile, F fixed.

$$0 = \mathcal{L}^2 = \mathcal{L}.M + \mathcal{L}.F$$

Lemma

M mobile $\Rightarrow$ M nef, $\mathcal{L}$ effective $\Rightarrow$ $M.\mathcal{L} \geq 0$.

$\mathcal{L}$ nef, F effective $\Rightarrow$ $\mathcal{L}.F \geq 0$.

$\Rightarrow$ $M.\mathcal{L} = \mathcal{L}.F = 0$. $\Rightarrow$ $M^2 + M.F = 0$

$F^2 + M.F = 0$.

M mobile $\Rightarrow M$ nef $\Rightarrow M^2 \geq 0$,

M mobile $\Rightarrow M$ nef, F effective $\Rightarrow M.F \geq 0$.

$\Rightarrow M^2 = M.F = 0 \Rightarrow F^2 = 0$ since $L^2 = 0$.

Then $h^0(F) + h^0(-F) = h^0(F) + h^2(F) \geq \chi(F) = 2 + \frac{F^2}{2} = 2$.

Since $F \geq 0 \Rightarrow h^0(-F) = 0$ for $F \neq 0$. $\Rightarrow h^0(F) \geq 2$.

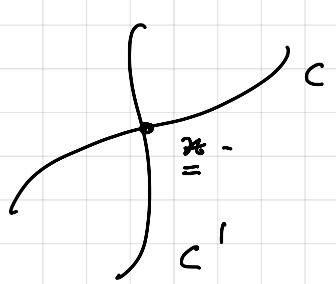
But we showed $h^0(F) = 1$. Thus $F = 0 \Rightarrow L = M$

$\Rightarrow Bs\,|L|$ dimension zero.

Step $\boxed{II}$ We show L basepoint free.

Let $x \in Bs\,|L|$, $C \in |L|$ fixed, $C' \in |L|$ arbitrary

$\Rightarrow C.C' = L^2 = 0$. Since $x \in C \cap C' \Rightarrow C, C'$ share



a component. If C irreducible

$\Rightarrow C \subseteq C' \;\; \forall\, C' \in |L| \Rightarrow$

$\Rightarrow C \subseteq Bs\,|L| =$ zero-dim'l. Contradiction.

Otherwise decompose C into components.

Step $\boxed{\text{III}}$ $\mathcal{L}$ basepoint free & $\phi : X \longrightarrow B \hookrightarrow \mathbb{P} H^0(\mathcal{L})$

$$\mathcal{L} = \phi^* \mathcal{O}_B(1).$$

B point $\Rightarrow$ L trivial false.

B surface $\Rightarrow$ $L^2 = \phi^* \mathcal{O}_B(1)^2 = \phi^*(\text{points}) > 0$

but $L^2 = 0$. false.

Thus B is a curve, reduced & irreducible.

Stein Factorization (H. III. 11.5)

$f : X \longrightarrow Y$ proper, $\exists$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & Z \arrow[d, "h"] \\ & Y \end{tikzcd}$$

(1) g proper surjective $g_* \mathcal{O}_X = \mathcal{O}_Z$ $\Rightarrow$ connected fibers

(2) h finite.

Construction

$$Z = \underline{\mathrm{Spec}}_{\mathcal{O}_Y} f_* \mathcal{O}_X \qquad f_* \mathcal{O}_X \to Y.$$

$X \xrightarrow{g} Z$ is natural and $g_* \mathcal{O}_X = \mathcal{O}_Z$ is immediate.

(Work affine locally).

If $Y = \mathrm{Spec}\, A$, $Z = \mathrm{Spec}\, H^0(X, \mathcal{O}_X) \Rightarrow g_* \mathcal{O}_X = \mathcal{O}_Z$.

f proper $\Rightarrow g$ proper

The statement that $g_* \mathcal{O}_X = \mathcal{O}_Z$ & g proper $\Rightarrow$ fibers are connected is Zariski's Connectedness Thm H. III. 11.

Claim X normal $\Rightarrow$ Z normal

Proof Work locally.

We show $\mathcal{O}_Z$ is integrally closed.

Let F be a rational function on Z with

$$F^n + a_1 F^{n-1} + \dots + a_n = 0, \quad a_j \in \mathcal{O}_Z. \Rightarrow F \in \mathcal{O}_Z.$$

$$\Rightarrow \quad g^* F^n + g^* a_1 \cdot g^* F^{n-1} + \dots + g^* a_n = 0., \quad g^* a_j \in \mathcal{O}_X.$$

X normal. $\Rightarrow g^* F \in \mathcal{O}_X. \Rightarrow F \in g_* \mathcal{O}_X = \mathcal{O}_Z.$

Let $\Phi : X \xrightarrow{g} \Sigma \xrightarrow{h} B$

$g_* \mathcal{O}_X = \mathcal{O}_\Sigma$, Σ normal, irreducible $\Rightarrow$ Σ smooth

---

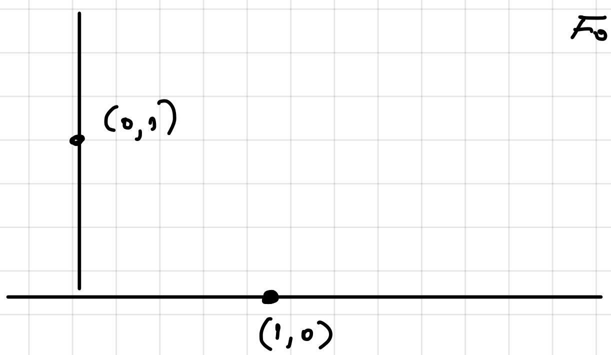
Claim $\Sigma \cong \mathbb{P}^1 \iff H^1(\Sigma, \mathcal{O}_\Sigma) = 0$.

Proof of claim Leray spectral sequence.

$$E_2^{i,j} = H^i(\Sigma, R^j g_* \mathcal{O}_X) \Rightarrow H^{i+j}(X, \mathcal{O}_X). = H^1(X, \mathcal{O}_X) = 0$$

$$E_2^{0,1} = H^0(\Sigma, R^1 g_* \mathcal{O}_X).$$

$$E_2^{1,0} = H^1(\Sigma, g_* \mathcal{O}_X) = H^1(\Sigma, \mathcal{O}_\Sigma).$$



Following the spectral sequence

$H^1(X, \mathcal{O}_X) = 0$ gives

$H^1(\Sigma, \mathcal{O}_\Sigma) = 0 \Rightarrow g(\Sigma) = 0$

$\Rightarrow \Sigma \cong \mathbb{P}^1$.

Conclusion We obtained $X \xrightarrow{g} \mathbb{P}^1$.

$$\mathcal{I} = \phi^* \mathcal{O}_B(1) = g^* h^* \mathcal{O}_B(1). \text{ Let } h^*\mathcal{O}_B(1) = \mathcal{O}_{\mathbb{P}^1}(m).$$

$$\Longrightarrow \mathcal{I} = g^* \mathcal{O}_{\mathbb{P}^1}(m) = \mathcal{O}(mf).$$

Generic fiber of g is:

- smooth by generic smoothness. H. III. 10.7

- connected. because Zariski connectedness.

$\Longrightarrow$ elliptic curve. since $F^2 = 0 = 2g(F) - 2 \Rightarrow g(F) = 1$.

Math 220B — Lecture 16

March 3, 2021

§1. Kodaira classification of singular fibers

Let X be a K3 and assume $X \longrightarrow \mathbb{P}^1$ elliptic fibration.

We saw (Lecture 7, page 5)

$$\sum_{X_b \text{ singular}} e(X_b) = 24.$$

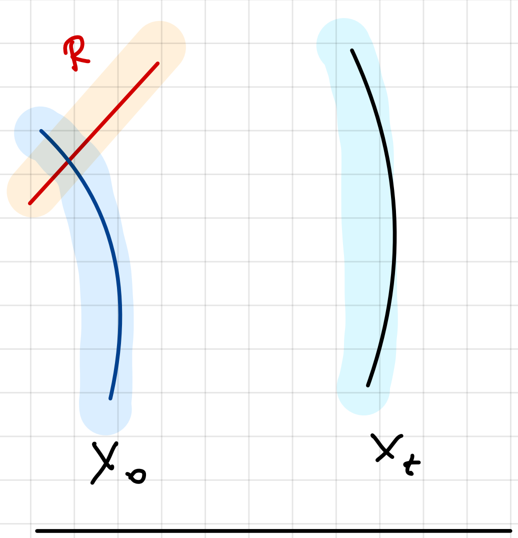
Question What are the singular fibers?

Useful remark

$R \subseteq X_0$ component of singular fiber X_0 $\longrightarrow$

$\Longrightarrow R \cdot X_0 = 0.$

$R \cdot X_0 = R \cdot X_t = 0.$



Zariski's Lemma + ε (Chp 11. Huybrechts).

$\pi : X \longrightarrow \mathbb{P}^1$ elliptically fibred K3 surface

$\boxed{i}$ fibers of π are connected

$\boxed{ii}$ $\nexists$ multiple fibers (they can be non-reduced).

$\boxed{iii}$ D supported on a fiber $\Rightarrow D^2 \leq 0$

$\boxed{iv}$ with equality iff $D = \alpha f$.

$\boxed{v}$ if a fiber is irreducible $\Rightarrow$ smooth, nodal or cuspidal

Proof

$\boxed{1}$ Let

- X_t = smooth connected fiber

- X_0 = singular fiber

$$0 \longrightarrow \mathcal{O}(-X_t) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X_t} \longrightarrow 0$$

$$H^0(X, \mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}_{X_t}) \longrightarrow H^1(\mathcal{O}(-X_t)). \longrightarrow H^1(\mathcal{O}) = 0$$

Since $\mathcal{O}(-X_t) = \mathcal{O}(-X_0)$ we have $H^1\,\mathcal{O}(-X_0) = 0$ so

$$0 \longrightarrow \underbrace{H^0(X, \mathcal{O})}_{\mathbb{C}} \xrightarrow{\sim} \underbrace{H^0(\mathcal{O}_{X_0})}_{\mathbb{C}} \longrightarrow H^1(\mathcal{O}(-X_0)) = 0.$$

$\Rightarrow$ $H^0(\mathcal{O}_{X_0}) = \mathbb{C}$ $\Rightarrow$ X_0 is connected.

$\boxed{ii}$ If $X_0 = mC$. Since $X_0^2 = 0 \Rightarrow C^2 = 0$, C connected

$$0 \longrightarrow \mathcal{O}((m-1)C) \longrightarrow \mathcal{O}(mC) \longrightarrow \mathcal{O}(mC)/_C \cong \mathcal{O}_C \longrightarrow 0$$

yields

$$\begin{array}{ccccccc} 0 \to H^0(\mathcal{O}((m-1)C)) & \longrightarrow & H^0\,\mathcal{O}(mC) & \longrightarrow & H^0(\mathcal{O}_C) \longrightarrow \\ & & \| & & \| \\ & & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) & & \mathbb{C} \\ & & \| & \xrightarrow{\neq 0} & \\ & & \mathbb{C}^2 & & \end{array}$$

$$\Rightarrow \quad h^0((m-1)C) = 1.$$

But $L = \mathcal{O}((m-1)C)$ satisfies $L^2 = 0$

Last time: Serre + HRR (Lecture 15, page 7)

$$h^0(L) + h^0(L^{-1}) = h^0(L) + h^2(L) \geq \chi(L) = 2$$

$$\Rightarrow \quad h^0(L) = 1 \;\; \& \;\; h^0(L^{-1}) \geq 1 \Rightarrow L \cong \mathcal{O} \Rightarrow m = 1$$

$\boxed{\text{iii}}$ If D supported on a fiber $\Rightarrow D^2 \leq 0$.

Assume

$$\left.\begin{array}{l} D^2 > 0 \\ D.f = 0 \end{array}\right. \Longrightarrow f^2 < 0 \quad \text{by Hodge index.}$$

This contradicts $f^2 = 0$.

useful remark above

$\boxed{\text{iv}}$ If $D^2 = 0$, we wish to show $D = \alpha f$.

Assume D is supported on a fiber X_0 and write

$$X_0 = \sum m_i R_i, \qquad D = \sum n_i R_i$$

Pick $\alpha \in \mathbb{Q}$ such that $D + \alpha f = \sum (n_i + \alpha m_i) R_i$

contains only positive coeff & negative coeff. This fails only if

$D = -\alpha f$.

Write $D + \alpha f = P - N$, $P, N > 0$.

$$\Rightarrow \quad (D + \alpha f)^2 = \underbrace{D^2}_{0} + 2\alpha \underbrace{D.f}_{0} + \alpha^2 f^2 = 0$$

by useful Remark.

$\quad //$

$$\Rightarrow \quad (P - N)^2 = 0$$

$\Rightarrow \quad 2\, P.N \le P^2 + N^2$. Now P, N are supported on

fibers so $P^2, N^2 \le 0$ by $\boxed{iii}$ & fibers are connected so

$P.N > 0$. This gives a contradiction.

$\boxed{iv}$ C is an irred fiber. Then Lecture 7, page 7:

$$\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 - p_a$$

$$= \chi(\mathcal{O}) - \chi(\mathcal{O}(-C)) = -\frac{C^2}{2} = 0.$$

$\Rightarrow \quad p_a = 1.$

(1) C smooth $\Rightarrow$ C elliptic

(2) C singular $\Rightarrow$ H. V. 3.9.2.

$$0 = g(\tilde{C}) = p_a(C) - \sum_{p \text{ sing}} \frac{1}{2} m_p(m_p - 1).$$

Since C is irred $\Rightarrow$ $\tilde{C}$ irred & $g(\tilde{C}) < 1$ so

$g(\tilde{C}) = 0$ & $\exists!$ unique point of multiplicity 2.

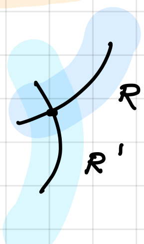
This can be either a node or a cusp.

Remark – X_0 fiber with ≥ 2 components

– R component $\Longrightarrow$ $R^2 = -2$. & $R \cong \mathbb{P}^1$.

Zariski

Indeed, $R^2 \leq 0$. If $R^2 = 0 \Rightarrow R = \alpha f \Rightarrow R.R' = \alpha f . R' = 0$ ↯



useful remark

Thus $R^2 < 0$ & $\chi(\mathcal{O}_R) = -\frac{R^2}{2} = 1 - p_a \leq 1$. Then

$R^2 = -2$ & $p_a = 0 \Longrightarrow R \cong \mathbb{P}^1$ (see Lecture 7, page 7)

Graph Let X_0 be a singular fiber

[i] vertices $\rightsquigarrow$ irred components of X_0

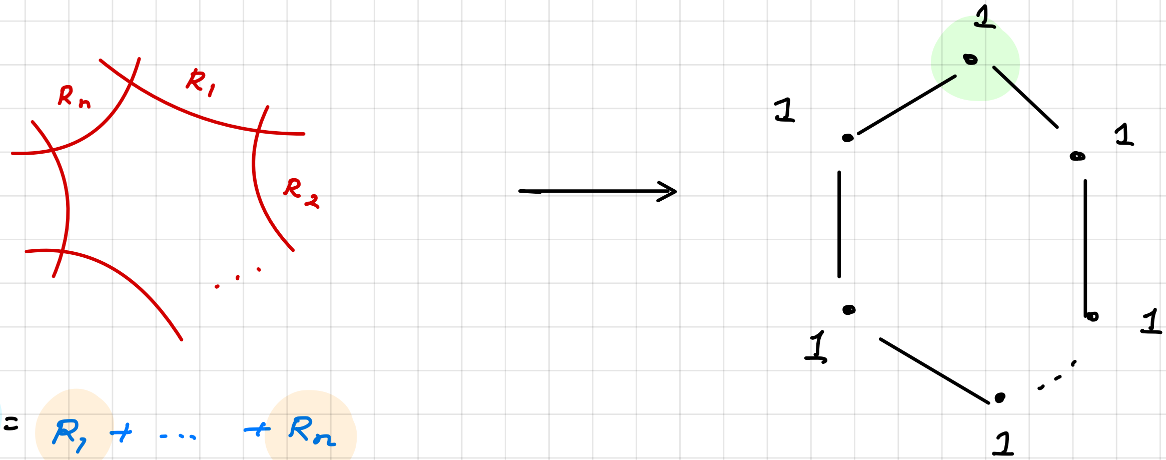
$\rightsquigarrow$ decorated by multiplicity

[ii] edges $\rightsquigarrow$ $R_i \cdot R_j$ edges between the corresponding vertices $i \neq j$

Strategy for classification

singular fiber $\Rightarrow$ graph $\Rightarrow$ quadratic form $\Rightarrow$ answer.

Example I_n



$\tilde{A}_{n-1}$

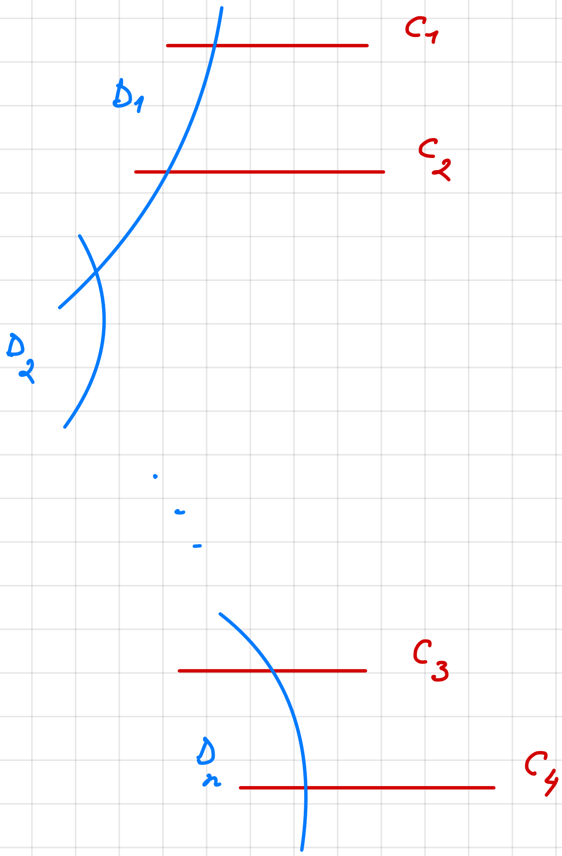
$X_0 = R_1 + \cdots + R_n$

n edges & vertices.

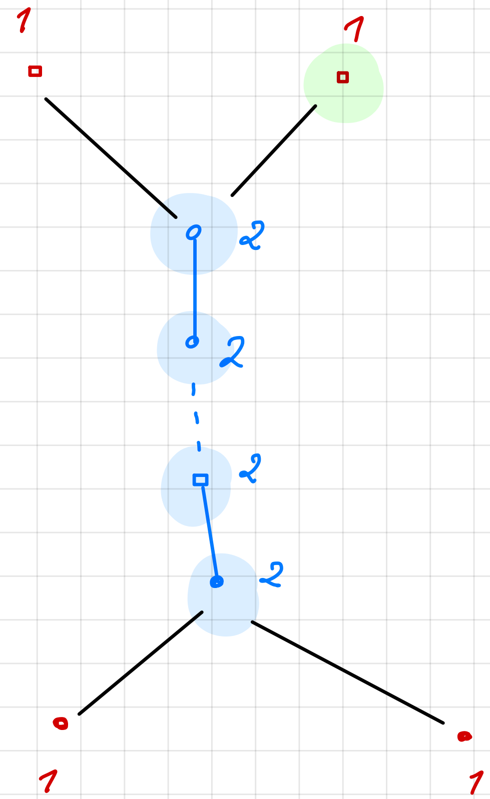
$e(X_0) = n.$

Example I_n^*

$$X_0 = C_1 + C_2 + C_3 + C_4 + 2(D_1 + \dots + D_n)$$



$\longrightarrow$



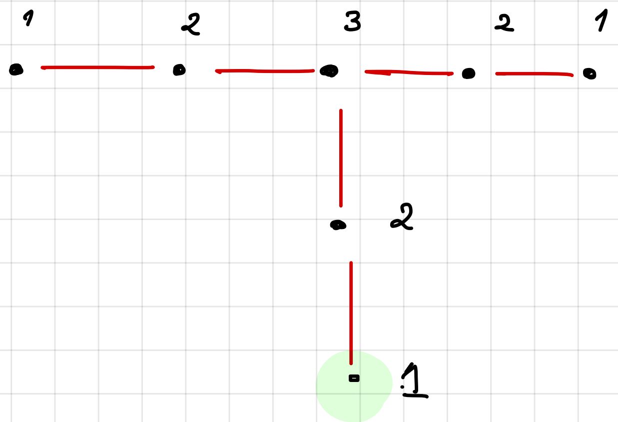
$\tilde{D}_{n+3} = n+4$ vertices

$$e(X_0) = n+6.$$

Other possibilities

$\tilde{E}_6$

$\overline{IV}^*$

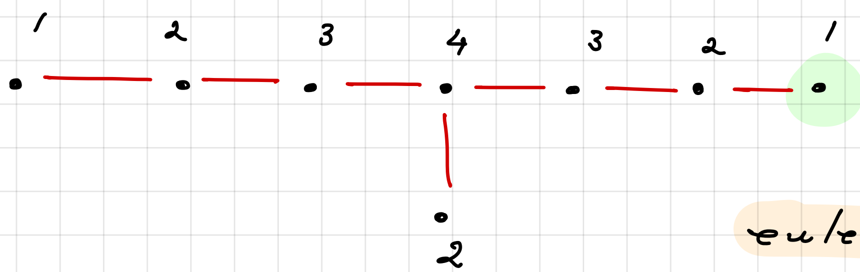


euler = 8

$T_{3,3,3}$

$\tilde{E}_7$

$\overline{III}^*$

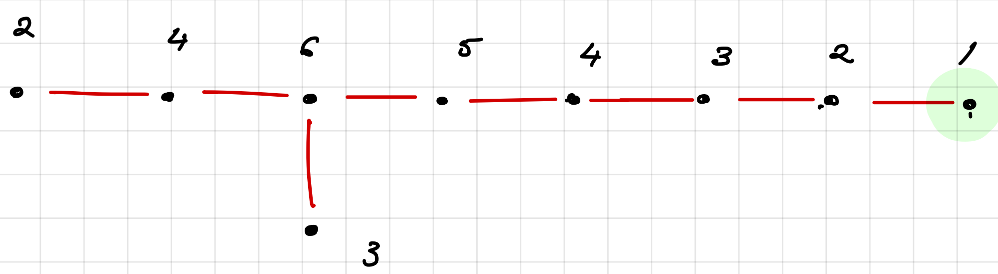


euler = 9

$T_{2,4,4}$.

$\tilde{E}_8$

$\overline{II}^*$

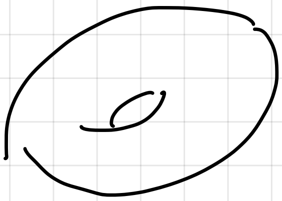


euler = 10.

$T_{2,3,6}$

Beware!

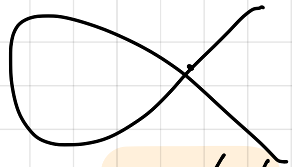
I_0



euler = 0

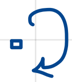
smooth.

I_1

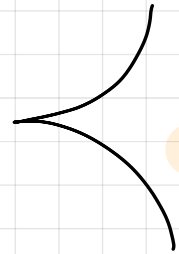


nodal

euler = 1

 $\tilde{A}_0$

$\overline{II}$.

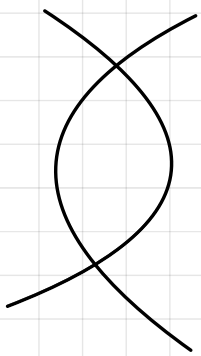


cuspidal.

euler = 2

1 vertex $\Rightarrow$

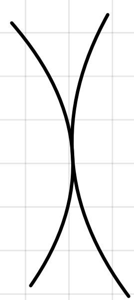
$\Rightarrow$ 1 component



I_2

euler = 2

$\overline{III}$

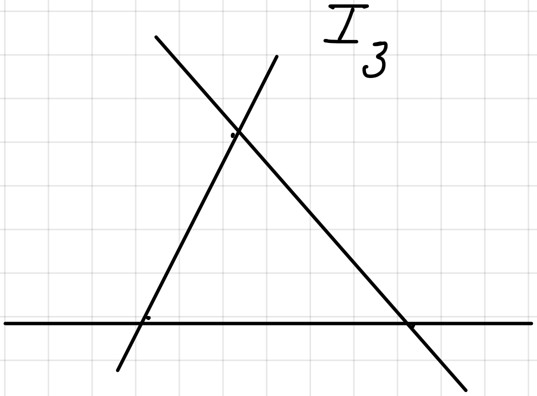


euler = 3

$\longrightarrow$

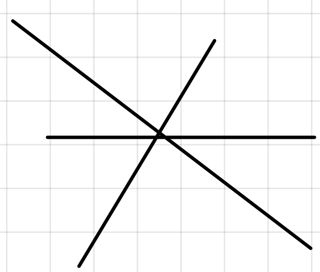
$\tilde{A}_1$

I_3



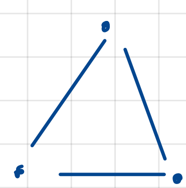
euler = 3

$\overline{IV}$.



euler = 4

$\longrightarrow$



$\tilde{A}_2$

It is not hard to check that all other graphs

$\tilde{A}$, $\tilde{D}$, $\tilde{E}$ uniquely determine the type of the fiber.

Since we discussed $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$ above, we assume

$$G \neq \tilde{A}_0, \tilde{A}_1, \tilde{A}_2$$

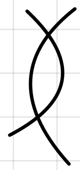
We show $G = \tilde{A}, \tilde{D}, \tilde{E}$.

Classification of fibers (not multiple)

(1) I_n including

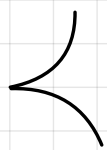
I_0 = smooth

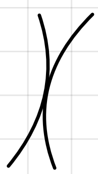
I_1 = nodal $\qquad \tilde{A}_{n-1}$

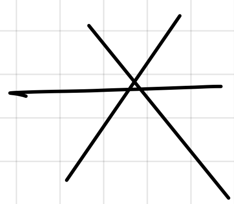
I_2 = 

...

(2) I_n^* $\qquad \tilde{D}_{n+3}$

(3) $\underline{\overline{II}}$ = cusp  $\qquad \tilde{A}_0$

$\underline{\overline{III}}$ =  $\qquad \tilde{A}_1$

$\underline{\overline{IV}}$ =  $\qquad \tilde{A}_2$

(4) $\underline{\overline{II}}^*$ $\qquad \tilde{E}_8$

$\underline{\overline{III}}^*$ $\qquad \tilde{E}_7$

$\underline{\overline{IV}}^*$ $\qquad \tilde{E}_6$

Strategy

Fiber $\Rightarrow$ Graph $\Rightarrow$ Quadratic Form $\Rightarrow$ Answer

Quadratic form

G graph, connected, vertices v_i

$$V = \bigoplus_i \mathbb{Q}\langle v_i \rangle$$

$$Q : V \times V \longrightarrow \mathbb{Q}\,, \quad Q(v_i, v_i) = -2$$

$$Q(v_i, v_j) = \#\text{ edges.}$$

This is slightly different for $\tilde{A}_0$ but we assumed $G \neq \tilde{A}_0$.

Example $G = \tilde{A}_{n-1} \implies V \cong \mathbb{Q}^n$

Quadratic Form =

$$Q\left(\sum_{i=1}^{n} x_i v_i\right) = -2\sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i x_{i+1}$$

$$= -\sum_{i=1}^{n} (x_i - x_{i+1})^2 \leq 0.$$

& $\operatorname{Ker} Q = \mathbb{Q}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$

Remark For a singular fiber, the quadratic form

$Q : V \times V \longrightarrow \mathbb{Q}$ satisfies

$\boxed{i}$ $Q \leq 0$

$\boxed{ii}$ kernel of Q is 1-dimensional, spanned by a vector with nonzero entries

Remark The same happens for the $\tilde{A}$, $\tilde{D}$, $\tilde{E}$ graphs.

Proposition Assume G is a connected graph with properties $\boxed{i}$ & $\boxed{ii}$ above. Then G is $\tilde{A}$, $\tilde{D}$, $\tilde{E}$

Proof $G \neq \tilde{A}_0, \tilde{A}_1$

Step 1 Any connected graph is contained or contains an extended Dynkin diagram. $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$.

Step 2 If G, G' are graphs as above & $G \leq G'$ then $G = G'$!

Proof – if $|G' \setminus G| \geq 1$,

$\operatorname{Ker} Q_G$ contains $\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix}$

$\operatorname{Ker} Q_{G'}$ contains $\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ & additional vector with non-zero entries

---

– G' contains no multiple edges. $\Rightarrow G = G'$!

since G, G' are connected.

If $\exists$ v, w vertices joined by $v.w \geq 2$ edges

then

$$Q(v+w) = v^2 + w^2 + 2\, v.w = -4 + 2\, v.w \geq 0.$$

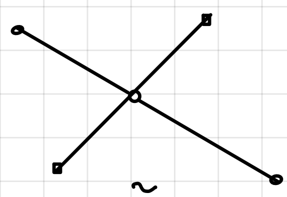
Thus $Q(v+w) = 0$ and $v+w$ spans $\operatorname{Ker} Q_{G'}$ $\Rightarrow$ G' has only 2 vertices & double edge $\Rightarrow \tilde{A}_1$. Contradiction.

Proof of Step 1

$\boxed{i}$ if $\exists$ loop in G. $\Rightarrow$ $\tilde{A} \subseteq G$.

$\boxed{ii}$ else G is a tree. Further discussion:

— $\exists$ vertex of valency ≥ 4. then

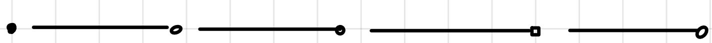


$\subseteq G$

$\tilde{D}_4$

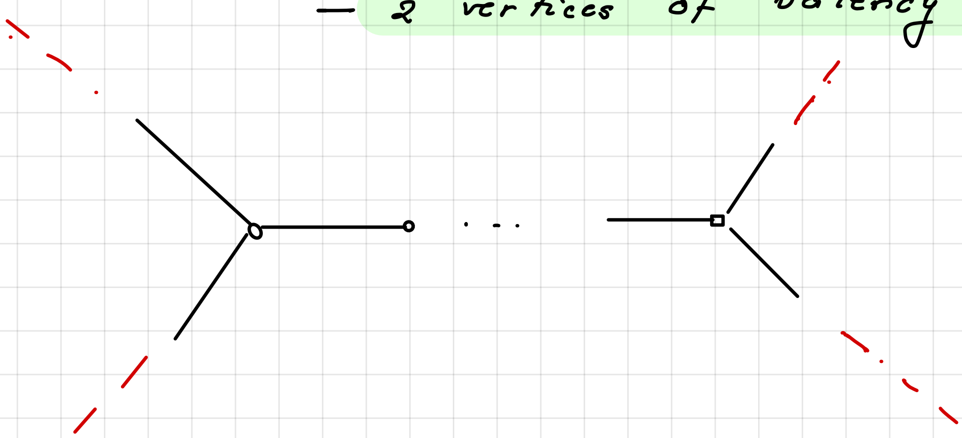
— if $\not\exists$ no vertex of valency 3, then

all vertices have valency 1 & 2. Then G is



$\Rightarrow$ $G \subseteq \tilde{A}$

— 2 vertices of valency $= 3$ then $\tilde{D} \subseteq G$.

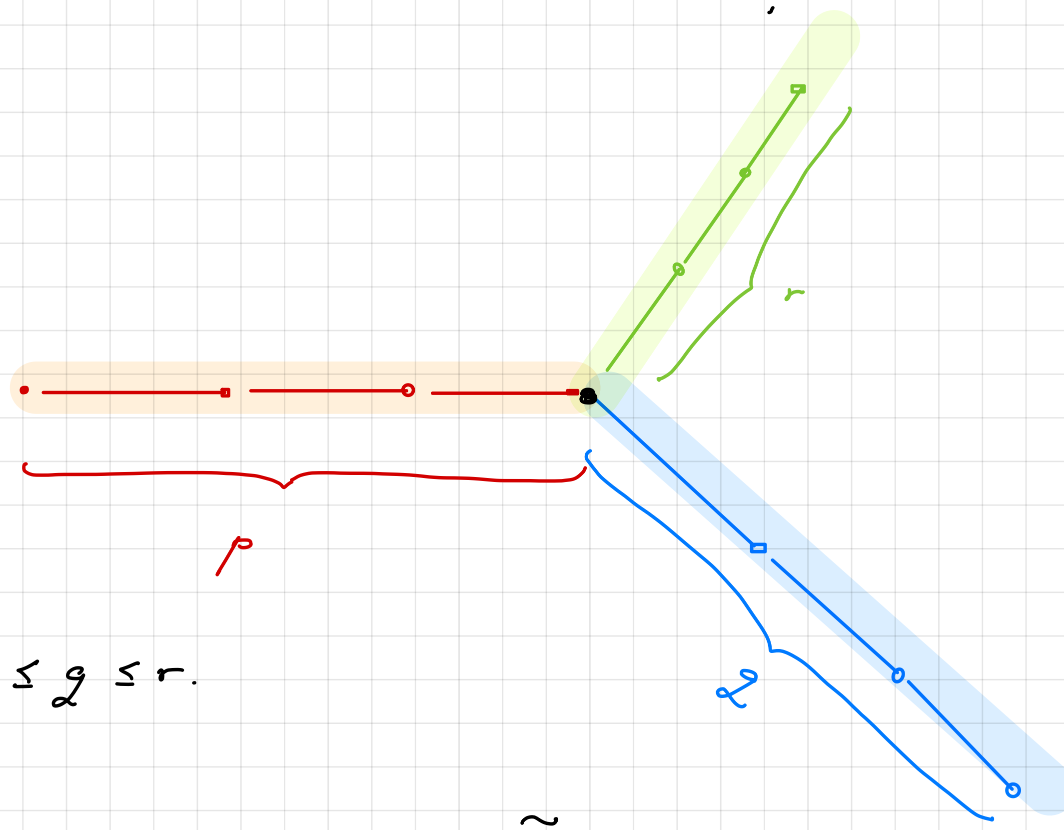


– in the remaining cases: one vertex of valency 3: T_{pqr}

$T_{3,3,3} = \tilde{E}_6$

$T_{2,4,4} = \tilde{E}_7$

$T_{2,3,6} = \tilde{E}_8$



Assume $2 \leq p \leq q \leq r$.

- $p \geq 3 \implies T_{3,3,3} \subseteq G \implies \tilde{E}_6 \subseteq G$
- $p = 2,\ q \geq 4 \implies T_{2,4,4} \subseteq G \implies \tilde{E}_7 \subseteq G$.
- $p = 2,\ q = 2 \implies G \subseteq \tilde{D}$
- $p = 2,\ q = 3,\ r \geq 6 \implies T_{2,3,6} \subseteq G \implies \tilde{E}_8 \subseteq G$
- $p = 2,\ q = 3,\ r \leq 6 \implies G \subseteq T_{2,3,6} \implies G \subseteq \tilde{E}_8$

Math 220B — Lecture 17

March 5, 2021

We discussed the classical geometry of (X, L) in some detail. We now turn to moduli spaces.

$$\mathcal{F}_g = \{(X, L) : L^2 = 2g-2, \ L \text{ primitive \& ample}\}/\sim$$

We have "seen" in Lecture 10 that $\mathcal{F}_g$ is a quotient

$$[H/PGL], \quad H \hookrightarrow Hilb$$

Huybrechts, V. shows H smooth & PGL acts with finite stabilizers

$\mathcal{F}_g$ is a smooth DM stack of dimension 19.

quasiprojective coarse moduli scheme $O(\Lambda_g) \backslash \mathcal{D}_g^\circ$

where $\mathcal{D}_g^\circ = \mathcal{D}_g \setminus \bigcup_{\substack{\delta^2=-2 \\ \delta \in \Lambda_g}} \delta^\perp$

Three moduli spaces

$\mathcal{M}_g$ = moduli of smooth curves

$\mathcal{M}_g, \mathcal{A}_g, \mathcal{F}_g$

$\mathcal{A}_g$ = ppav. $(A, \mathcal{L})$

$\mathcal{F}_g$ = moduli of K3s. $(X, \mathcal{L})$

$t : \mathcal{M}_g \longrightarrow \mathcal{A}_g$

$\dim \mathcal{M}_g = 3g-3$, $\quad \mathcal{M}_g = [H/PGL]$.

$\dim \mathcal{A}_g = \frac{g(g+1)}{2}$, $\quad \mathcal{A}_g = [H/PGL]$.

$\dim \mathcal{F}_g = 19$, $\quad \mathcal{F}_g = [H/PGL]$.

In all cases $H \hookrightarrow \mathrm{Hilb}(\mathbb{P})$, the Hilbert scheme of a suitable projective space.

We use $\mathcal{L}^{\otimes 3}$ to embed into projective space in the case of $\mathcal{A}_g$ & $\mathcal{F}_g$. For curves we can use $K_C^{\otimes m}$ = very ample, $m \geq 3$

$$C \longrightarrow \mathbb{P} H^0(K_C^{\otimes m}) \cong \mathbb{P} V.$$

Note that $\mathcal{F}_g = O \backslash \mathcal{D}^\circ$ where $\mathcal{D}^\circ \underset{\text{open}}{\hookrightarrow}$ type $\underline{IV}$ domain.

Now, $\mathcal{A}_g$ can be described as

$$Sp(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$$

where

$$\mathfrak{H}_g = \{ \Omega \in Mat_{\mathbb{C}}(g \times g) : \Omega = \Omega^t, \ Im\, \Omega > 0 \}.$$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp \text{ acts } M\Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

In this description, each Ω determines a torus

$$X_\Omega = \mathbb{C}^g / \Omega \mathbb{Z}^g + \mathbb{Z}^g$$

& the Riemann relations

$\Omega = \Omega^t$, $Im\, \Omega > 0$ ensure X_Ω is abelian variety.

with polarization induced by Ω^{-1}.

There is a map

$$t : M_g \longrightarrow A_g \qquad C \longmapsto (Jac(C), \Theta).$$

$$Jac(C) = H^0(\omega_C)^\vee \big/ H_1(C, \mathbb{Z}) = V/\Gamma$$

$$\gamma \in H_1(C, \mathbb{Z}) \rightsquigarrow \left\{ \ell_\gamma : \omega \longmapsto \int_\gamma \omega \right\}.$$

Note

$$H^0(Jac(C), \Omega^1_{Jac(C)}) = (1,0) \text{ forms on } V.$$

$$= V^\vee = H^0(\omega_C)^{\vee\vee} = H^0(C, \omega_C).$$

The theta divisor can be described geometrically as

$\Theta = \{ L : h^0(L \otimes M) \neq 0 \} \hookrightarrow Jac^0(C)$ where M is a line bundle of degree $g-1$. This is uniquely defined only up to translations.

The Picard Groups

(1) $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g) = \langle \lambda \rangle$ $g \geq 3$ ↙ Harer (topological).

(2) $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_g) = \langle \lambda \rangle$ $g \geq 3$ ↙ Borel (arithmetic)

(3) $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_g) \longrightarrow \infty$ as $g \to \infty$. ↙ O'Grady (geometry).

known ↙ Borcherds – Bruinier

generators ↙ Millson - Li – Bergeron – Moeglin

Hodge Bundles The moduli spaces $\mathcal{M}_g$, $\mathcal{A}_g$, $\mathcal{F}_g$ carry

Hodge bundles $\mathbb{E}_g^{M}$, $\mathbb{E}_g^{A}$, $\mathbb{E}_g^{K}$

$$\begin{array}{ccc} \mathbb{E}_g^{M} & & H^0(\omega_C) \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \ni & C \end{array} \qquad \begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ \mathcal{M}_g \end{array} \qquad \begin{array}{l} \mathbb{E}_g^{M} = \pi_* \omega_\pi \\ \operatorname{rk} \mathbb{E}_g^{M} = g. \end{array}$$

$$\begin{array}{ccc} \mathbb{E}_g^{A} & & H^0(\Omega^1_A). \\ \downarrow & & \downarrow \\ \mathcal{A}_g & \ni & (A, L) \end{array} \qquad \begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathcal{A}_g \end{array} \qquad \operatorname{rk} \mathbb{E}_g^{A} = g.$$

$$\begin{array}{ccc} \mathbb{E}_g^{K} & & H^0(\Omega^2_X) \\ \downarrow & & \downarrow \\ \mathcal{F}_g & \ni & (X, L) \end{array} \qquad \begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathcal{F}_g \end{array} \qquad \operatorname{rk} \mathbb{E}_g^{K} = 1.$$

Remark $t : \mathcal{M}_g \longrightarrow \mathcal{A}_g$, $\quad t^* \mathbb{E}_g^{A} = \mathbb{E}_g^{M}$.

$$\lambda_i^{M} = c_i(\mathbb{E}_g^{M}), \quad 1 \leq i \leq g$$

$$\lambda_i^{A} = c_i(\mathbb{E}_g^{A}), \quad 1 \leq i \leq g$$

$$\lambda = c_1(\mathbb{E}_g^{K}).$$

κ - classes We can define these over the three spaces

$$\kappa_i^{M} = \pi_* c_1(\omega_\pi)^{i+1}$$

$$\kappa^{A}_{i,j,\ell} = \pi_* c_i(\Omega^j_\pi)^{\ell+1}$$

$$\kappa_i^{K3} = \pi_* c_2(\Omega^1_\pi)^{i+1}$$

Remark In the case of $\mathcal{A}_g$, all κ - classes are zero.

Indeed, $H^0(A, \Omega_A) \otimes \mathcal{O}_A \longrightarrow \Omega_A$ isomorphism. Thus

$$\pi^* \pi_* \Omega^1_\pi \longrightarrow \Omega^1_\pi \Rightarrow \Omega^1_\pi \cong \pi^* \mathbb{E}$$

$$\Rightarrow \kappa_j = \pi_* c_j(\Omega^1_\pi) = \pi_* c_j(\pi^* \mathbb{E}) = c_j(\mathbb{E}) \underbrace{\pi_* 1}_{0} = 0.$$

Remark For $\mathcal{F}_g$, we have $\Omega^2_\pi \cong \pi^* \mathbb{E}$

The only reasonable choice in the definition of κ - classes

above is $\pi_* c_2(\Omega^1_\pi)^{i+1}$.

The tautological rings (1st attempt).

$R^*(M_g)$, $R^*(\mathcal{A}_g)$, $R^*(\mathcal{F}_g)$ is the subring of Chow generated by both λ & κ - classes.

$$R^* \cong \mathbb{C}[\lambda, \kappa] / \text{Relations}$$

Thm: $R^* = \bigoplus_{k=0}^{d} R^k$ satisfies Poincaré duality if

(1) $R^d \cong \mathbb{C}$.

(2). $\forall k$: $\quad R^k \times R^{d-k} \xrightarrow{\cdot} R^d \cong \mathbb{C}$ perfect pairing.

Question

Do the rings

$$R^*(\mathcal{M}_g),\ R^*(\mathcal{A}_g),\ R^*(\mathcal{F}_g) \text{ satisfy Poincaré duality?}$$

Question

How do we get relations between generators?

Common Features

(1) in all three cases, we will get relations via GRR for the universal family & natural bundles (trivial, etc).

(2) for $\mathcal{A}_g$ and $\mathcal{F}_g$ we will describe these rings completely.

(3) Applying GRR we will obtain

For $\mathcal{M}_g \rightsquigarrow$ Mumford relation

$\mathcal{A}_g \rightsquigarrow \lambda_g = 0$ + Mumford

$\mathcal{F}_g \rightsquigarrow$ we also get relations (van der Geer).

Grothendieck Riemann-Roch

$\pi : \mathcal{X} \longrightarrow \mathcal{Y}$. $\mathcal{V} \longrightarrow \mathcal{X}$ vector bundle

$$R\pi_! \mathcal{V} = \sum (-1)^k R^k \pi_* \mathcal{V}.$$

GRR $\quad ch(R\pi_! \mathcal{V}) = \pi_! (ch \mathcal{V} \cdot Todd(T^\pi))$

For a bundle B, $\quad Todd(B) = \prod_{k=1}^{rk} \frac{b_i}{1 - e^{-b_i}}$ $\}$ Chern roots of B.

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_k (-1)^{k-1} \frac{B_{2k}}{2k!} x^{2k}.$$

The case of $\mathcal{M}_g$ (Mumford) The simplest case:

$$\pi : \mathcal{C}_g \longrightarrow \mathcal{M}_g , \quad \mathcal{V} = \mathcal{O}_{\mathcal{C}} .$$

Serre duality

- $$\pi_! \, \mathcal{O}_{\mathcal{C}} = \pi_* \mathcal{O}_{\mathcal{C}} - R^1 \pi_* \mathcal{O}_{\mathcal{C}} = \mathbb{C} - \mathbb{E}^{\vee}$$

- $$\mathrm{ch}\, \pi_! \, \mathcal{O}_{\mathcal{C}} = 1 - \mathrm{ch}\, \mathbb{E}^{\vee}$$

$$= \pi_* \left(\mathrm{ch}\, \mathcal{O}_{\mathcal{C}} \cdot \mathrm{Todd}^{rel}(T^{rel}) \right).$$

$$= \pi_* \, \mathrm{Todd}(T^{rel}) = \pi_* \frac{-\omega}{1 - e^{\omega}} .$$

where $\omega = c_1(\omega_\pi)$.

$$\Rightarrow \quad 1 - \mathrm{ch}\, \mathbb{E}^{\sim} = \pi_* \left(\frac{-\omega}{1 - e^{\omega}} \right)$$

$$= \pi_* \left(1 - \frac{\omega}{2} + \sum (-1)^{k-1} \frac{B_{2k}}{2k!} \omega^{2k} \right).$$

$$= 1 - g + \sum_{k \geq 1} (-1)^{k-1} \frac{B_{2k}}{(2k)!} \kappa_{2k-1} .$$

$$\Rightarrow \quad \mathrm{ch}\, \mathbb{E} = g + \sum_{k \geq 1} (-1)^{k-1} \frac{B_{2k}}{(2k)!} \kappa_{2k-1} .$$

Lemma

$$c(W) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} (k-1)!\, ch_k(W) \right)$$

Proof Both sides are multiplicative as $W \rightsquigarrow W_1 + W_2$

Suffices to assume $W = L$. $=$ rank 1. Write $c_1(L) = \ell$

We show $1 + \ell = \exp\left(\sum_{k \geq 1} (-1)^{k-1} (k-1)!\, \frac{\ell^k}{k!} \right)$

$$= \exp\left(\sum_{k \geq 1} (-1)^{k-1} \frac{\ell^k}{k} \right) = \exp \log(1+\ell) = 1 + \ell. \checkmark$$

In our case, this yields

$$c(E) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_{2k}}{(2k)(2k-1)} K_{2k-1} \right)$$

$$c(E^\vee) = \exp\left(- \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_{2k}}{2k(2k-1)} K_{2k-1} \right)$$

$$\Rightarrow c(E)\, c(E^\vee) = 1.$$

Conclusion (Mumford relation)

(1) Hodge classes are in the span of the κ's.

In particular $R^*(M_g)$ is generated by κ's.

(2) $c(\mathbb{E})\, c(\mathbb{E}^\vee) = 1.$

$$\Rightarrow \quad (1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots \quad) = 0.$$

$$\Rightarrow \quad \lambda_g^2 = 0 \text{ over } M_g$$

The Hodge bundle of M_g also extends to $\overline{M}_g$ with the same expression. The Mumford relations extend over $\overline{M}_g$.

Known Facts

(1) $R^{>g-2}(M_g) = 0$ Looijenga

(2) $R^{=g-2}(M_g) = \mathbb{C}$ Faber

Diaz

(3) all complete subvarieties in M_g have $\dim \leq g-2$

(1) $\Longrightarrow$ (3)

If Z is complete of dimension $g-1$ (or higher) then

$\lambda_1^{g-1}/_Z \neq 0$ since λ_1 is known to be ample. (Baily - Borel)

But $\lambda_1^{g-1} \in R^{g-1}(M_g) = 0$ by (1).

Remark $\lambda_g \lambda_{g-1} = 0$ over $\partial \overline{M}_g$ (Faber)

(1) irreducible $\overline{M}_{g-1,2} \xrightarrow{\pi} \overline{M}_g$. Check

$$0 \longrightarrow \mathbb{E}_{g-1} \longrightarrow \pi^* \mathbb{E}_g \longrightarrow \mathcal{O} \longrightarrow 0 \quad \text{gives}$$

$\pi^* \lambda_g = 0$ by taking Chern classes. $\Rightarrow \pi^* \lambda_g \lambda_{g-1} = 0$

(2) reducible case $i : \overline{M}_{h,1} \times \overline{M}_{g-h,1} \longrightarrow \overline{M}_g$. Check.

$$i^* \mathbb{E}_g = \mathbb{E}_h^{(1)} + \mathbb{E}_{g-h}^{(2)}. \quad \text{Then}$$

$$i^* \lambda_g = \lambda_h^{(1)} \lambda_{g-h}^{(2)}$$

$$i^* \lambda_{g-1} = \lambda_h^{(1)} \lambda_{g-h-1}^{(2)} + \lambda_{h-1}^{(1)} \lambda_{g-h}^{(2)}$$

$$\Rightarrow i^* (\lambda_g \lambda_{g-1}) = \underbrace{\left(\lambda_h^{(1)}\right)^2}_{0} \lambda_{g-h}^{(2)} \lambda_{g-h-1}^{(2)} + \lambda_h^{(1)} \lambda_{h-1}^{(1)} \underbrace{\left(\lambda_{g-h}^{(2)}\right)^2}_{0} = 0$$

using the Mumford relation over $\overline{M}_{h,1}$ & $\overline{M}_{g-h,1}$.

$$\varepsilon : R^{g-2}(M_g) \longrightarrow \mathbb{C}, \qquad \varepsilon(\alpha) = \int_{\overline{M}_g} \alpha \lambda_g \lambda_{g-1} \quad \text{well-defined.}$$

Math 220B — Lecture 18

March 10, 2021

§ 0. Last time

Hodge Bundles The moduli spaces $\mathcal{M}_g$, $\mathcal{A}_g$, $\mathcal{F}_g$ carry Hodge bundles

$$\begin{array}{ccccc} \mathbb{E}_g^M & & H^0(\omega_C) & \qquad & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \mathcal{M}_g & \ni & C & & \mathcal{M}_g \end{array} \qquad \begin{array}{l} \mathbb{E}_g^M = \pi_* \omega_\pi \\ \operatorname{rk} \mathbb{E}_g^M = g. \end{array}$$

$$\begin{array}{ccccc} \mathbb{E}_g^A & & H^0(\Omega_A^1). & \qquad & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \mathcal{A}_g & \ni & (A, L) & & \mathcal{A}_g \end{array} \qquad \operatorname{rk} \mathbb{E}_g^A = g.$$

$$\begin{array}{ccccc} \mathbb{E}_g^K & & H^0(\Omega_X^2) & \qquad & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \mathcal{F}_g & \ni & (X, L) & & \mathcal{F}_g \end{array} \qquad \operatorname{rk} \mathbb{E}_g^K = 1.$$

$$\lambda_i^M = c_i(\mathbb{E}_g^M), \quad 1 \leq i \leq g$$

$$\lambda_i^A = c_i(\mathbb{E}_g^A), \quad 1 \leq i \leq g$$

$$\lambda = c_1(\mathbb{E}_g^K).$$

$$\kappa_i^M = \pi_* c_1(\omega_\pi)^{i+1}$$

$$\kappa_i^A = 0$$

$$\kappa_i^{K3} = \pi_* c_2(\Omega_\pi)^{i+1}$$

In all three cases, we defined

$$R^* = \mathbb{Q}[\lambda, \kappa] / \text{Relations}.$$

For M_g, we applied GRR to

$\pi : \mathcal{C} \longrightarrow M_g$ & the trivial sheaf $\mathcal{O}_{\mathcal{C}}$.

(1) only κ - classes are needed to generate

λ's can be expressed in terms of κ.

(2) Mumford relation $c(\mathbb{E})\, c(\mathbb{E}^\vee) = 1$.

$\Longleftrightarrow$ $(1 + \lambda_1 + \lambda_2 + \dots + \lambda_g)(1 - \lambda_1 + \lambda_2 - \dots \pm \lambda_g) = 1$.

$$R^*(M_g) = \mathbb{Q}[\kappa_1, \dots] / \text{Relations}.$$

Next – we apply the same idea to A_g & $\mathcal{F}_g$.

Goal – we will give an explicit description of

$R^*(A_g)$ and $R^*(\mathcal{F}_g)$.

Over A_g we give two relations

(1) $\lambda_g = 0$ GRR $\pi : \mathcal{X} \longrightarrow A_g$, $\mathcal{O}_{\mathcal{X}}$.

(2) Mumford relation GRR. $\pi : \mathcal{X} \longrightarrow A_g$, $\mathcal{L}$.

Over $\mathcal{F}_g$ we give relations

(1) κ – classes are powers of λ GRR

$\pi : \mathcal{X} \longrightarrow \mathcal{F}_g$, $\mathcal{O}_{\mathcal{X}}$.

(2) $\lambda^{18} = 0$ arithmetic techniques + Mumford.

Outcome

- abelian $R^*(A_g) \cong \mathbb{Q}[\lambda_1, \dots \lambda_g] \big/ (\lambda_g = 0, \text{ Mumford relation})$

- K3 $R^*(\mathcal{F}_g) \cong \mathbb{Q}[\lambda] \big/ \lambda^{18}$

---

§ 1. Abelian varieties

Theorem Over A_g, we have the following

$\boxed{i}$ $\lambda_g = 0$

$\boxed{ii}$ Mumford relation $c(\mathbb{E})\, c(\mathbb{E}^\vee) = 1$.

$$(1 + \lambda_1 + \lambda_2 + \dots + \lambda_g)(1 - \lambda_1 + \lambda_2 - \dots \pm \lambda_g) = 1$$

Proof of II Apply GRR to the universal family

$\pi : \mathcal{X} \longrightarrow \mathcal{A}_g$ and $\mathcal{O}_{\mathcal{X}}$.

$$\mathrm{ch}\ \pi_! \mathcal{O}_{\mathcal{X}} \underset{\text{GRR}}{=} \pi_* \left(\mathrm{ch}\ \mathcal{O}_{\mathcal{X}} \cdot \mathrm{Todd}(T^\pi) \right) = \pi_* \mathrm{Todd}(T^\pi).$$

$$H^0(A, \Omega_A^1) \otimes \mathcal{O}_A \xrightarrow{\sim} \Omega_A^1 \implies \pi^* \underbrace{\pi_* \Omega_\pi} \xrightarrow{\sim} \Omega_\pi$$

$$\implies \pi^* \mathbb{E} \xrightarrow{\sim} \Omega_\pi \implies T^\pi \cong \pi^* \mathbb{E}^\vee.$$

$$\implies \pi_* \mathrm{Todd}(T^\pi) = \pi_* \mathrm{Todd}(\pi^* \mathbb{E}^\vee) = \mathrm{Todd}(\mathbb{E}^\vee) \cdot \underbrace{\pi_* 1}_{0} = 0$$

$$\pi_! \mathcal{O}_{\mathcal{X}} = \sum_{k=0}^{g} (-1)^k R^k \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O} - \mathbb{F} + \Lambda^2 \mathbb{F} - \Lambda^3 \mathbb{F} + \dots$$

where $R^1 \pi_* \mathcal{O}_{\mathcal{X}} = \mathbb{F} \longrightarrow \mathcal{A}_g$ rank g.

Then $R^k \pi_* \mathcal{O}_{\mathcal{X}} = \Lambda^k \mathbb{F}$.

because $H^k(A, \mathcal{O}) = \Lambda^k H^1(A, \mathcal{O})$ for any abelian variety A.

$$\implies \mathrm{ch}\left(\mathcal{O} - \mathbb{F} + \Lambda^2 \mathbb{F} - \Lambda^3 \mathbb{F} + \dots \right) = 0 \qquad (*).$$

Key Formula

$$\mathrm{ch}\,\Lambda_{-1} W^\vee = c_{top}(W)\ \mathrm{Todd}(W)^{-1} \Rightarrow$$

Proof

Both sides are multiplicative $W \longrightarrow W_1 + W_2$. For LHS, note

$$\Lambda_{-1} W = \Lambda_{-1} W_1 \otimes \Lambda_{-1} W_2 \Longleftrightarrow \Lambda^k W = \bigoplus_{i+j=k} \Lambda^i W_1 \otimes \Lambda^j W_2 .$$

Thus $\mathrm{ch}\,\Lambda_{-1} W = \mathrm{ch}\,\Lambda_{-1} W_1 \cdot \mathrm{ch}\,\Lambda_{-1} W_2$. The right hand side:

$$c_{top}(W) = c_{top}(W_1)\, c_{top}(W_2), \qquad \mathrm{Todd}(W) = \mathrm{Todd}(W_1)\,\mathrm{Todd}(W_2)$$

By splitting principle we may assume $W = L$, $c_1(L) = \ell$.

$$\Lambda_{-1} L^\vee = \mathcal{O} - L^\vee \Rightarrow$$

$$\mathrm{ch}\,\Lambda_{-1} L^\vee = 1 - e^{-\ell} = \ell \cdot \left(\frac{\ell}{1-e^{-\ell}}\right)^{-1} = c_{top}(L) \cdot \mathrm{Todd}(L)^{-1}$$

By $(*)$ we obtain $c_g(F^\vee)\ \mathrm{Todd}(F^\vee)^{-1} = 0 \Rightarrow c_g(F^\vee) = 0$.

Remark Let $i : A \longrightarrow A^t$. Then $i^* \mathbb{F} = \mathbb{E}^\vee$. Indeed

$$H^1(A, \mathcal{O})^\vee = H^0(A^t, \Omega^1_{A^t}).$$

This can be seen writing

$$A = V/\Gamma \quad , \quad A^t = V^t/\Gamma^t \quad , \quad V^t = \mathrm{Hom}_{\text{anti}}(V, \mathbb{C}).$$

$$\Gamma^t = \mathrm{Hom}(\Gamma, \mathbb{Z}).$$

$$H^1(A, \mathcal{O}_A) = V^t$$

$$H^0(A^t, \Omega_{A^t}) = V^{t\vee}.$$

Then $c_{top}(\mathbb{F}^\vee) = 0 \implies c_{top}(\mathbb{E}) = 0 \implies \lambda_g = 0$

The Mumford Relation

GRR to universal polarization $\mathcal{L}$.

$$\begin{array}{c} \mathcal{X},\ \mathcal{L} \\ \downarrow \\ A_g \end{array}$$

Let $z : A_g \longrightarrow \mathcal{X}$ be the zero section.

WLOG $\mathcal{L}/_z \cong \mathcal{O}_z$.

Claim $\pi_! \mathcal{L}^{\otimes n} = \pi_! \mathcal{L} \otimes R$ where R vector space of dim n^g over a finite cover of A_g.

$\Longrightarrow \quad \pi_* \mathcal{L}^n = \pi_* \mathcal{L} \otimes R.$

(Ben Moonen, Chp 13)

GRR computation

$$\pi_! \mathcal{L}^n = \pi_! \mathcal{L} \otimes R \qquad \dim R = n^g.$$

- $\mathcal{L}$ principal $\Rightarrow h^0(A, \mathcal{L}) = 1$, $h^i(A, \mathcal{L}) = 0 \quad \forall\, i > 0$.

$\Rightarrow \pi_! \mathcal{L} = \pi_* \mathcal{L}$. = line bundle

$\Rightarrow c_1(\pi_! \mathcal{L}) = \theta \quad \Rightarrow \operatorname{ch} \pi_! \mathcal{L} = e^{\theta}$.

$$\Longrightarrow \quad \mathrm{ch}\, \pi_! \mathcal{L}^n = n^g \, \mathrm{ch}\, \pi_! \mathcal{L} = n^g e^{\theta} \qquad c_1(\mathcal{L}) = \ell.$$

$$\| \text{ GRR}$$

$$\pi_* \left(\mathrm{ch}\, \mathcal{L}^n \cdot \mathrm{Todd}\,(T^\pi) \right) = \pi_* \left(e^{n\ell} \cdot \pi^* \mathrm{Todd}\,(\mathbb{E}^\vee) \right)$$

$$= \pi_* (e^{n\ell}) \cdot \mathrm{Todd}\,(\mathbb{E}^\vee).$$

$$\Longrightarrow \quad \pi_* \left(\sum_{k \geq 0} \frac{n^{g+k} \ell^{g+k}}{(g+k)!} \right) \cdot \mathrm{Todd}\,(\mathbb{E}^\vee) = n^g e^{\theta} \quad \forall n$$

$$\Longrightarrow \quad \pi_* \left(\frac{\ell^{g+k}}{(g+k)!} \right) \cdot \mathrm{Todd}\,(\mathbb{E}^\vee) = 0 \quad \text{for } k \neq 0$$

$$\underbrace{\pi_* \left(\frac{\ell^g}{g!} \right)}_{1} \mathrm{Todd}\,(\mathbb{E}^\vee) = e^{\theta} \quad \text{for } k = 0$$

$$\Longrightarrow \quad \mathrm{todd}\, \mathbb{E}^\vee = e^{\theta}.$$

Let $\alpha_1 \ldots \alpha_g$ be the roots of $\mathbb{E}^\vee$.

$$\Longrightarrow \quad \prod_j \frac{\alpha_j}{1 - e^{-\alpha_j}} = e^{\theta} \Longrightarrow \prod_j \left(1 + \frac{\alpha_j}{2} + \cdots \right) = e^{\theta} = 1 + \theta + \frac{\theta^2}{2} + \cdots$$

$$\Longrightarrow \quad \theta = \sum \alpha_j / 2. \quad \Longrightarrow \quad \prod_j \frac{\alpha_j}{1 - e^{-\alpha_j}} = e^{\sum_j \alpha_j / 2}$$

$$\Rightarrow \prod_j \underbrace{\frac{\alpha_j}{e^{\alpha_j/2} - e^{-\alpha_j/2}}}_{\text{even function in } \alpha_j} = 1$$

$\Rightarrow$ all power sums in α_j^2 are zero .

$\Rightarrow$ all elementary symm. functions in α_j^2 are zero

$\Rightarrow \prod_j (1 - \alpha_j^2) = 1 \Rightarrow c(\mathbb{E})\, c(\mathbb{E}^\vee) = \prod (1+\alpha_j)(1-\alpha_j) = 1$

$\Rightarrow$ Mumford relation.

The ring R_g Define

$$R_g = \mathbb{Q}[u_1, \dots u_g] \Big/ \text{Mumford } (1+u_1+\dots+u_g)(1-u_1+u_2-\dots)=1 .$$

Note $R_g / u_g \cong R_{g-1}$

We obtain $R_{g-1} \twoheadrightarrow R^*(A_g)$. surjective

$$u_i \longrightarrow \lambda_i$$

Theorem A This is an isom. $R^*(\mathcal{A}_g) \cong R_{g-1}$

Moreover, $R^*(\mathcal{A}_g)$ satisfies Poincaré duality.

$$\Uparrow$$

Theorem B $\lambda_1^{g(g-1)/2} \neq 0$ in $\mathcal{A}_g$.

Proof of Thm A

Consider the ring $R_g = \mathbb{Q}[u_1, \ldots u_g] / \text{Mumford}$.

$$(1 + u_1 + u_2 + \cdots + u_g)(1 - u_1 + u_2 - \cdots \pm u_g) = 1. \quad (*)$$

Claim $u_k^2 u_{k+1} \cdots u_g = 0$

Indeed, from $(*)$ we find $u_g^2 = 0$.

Also $u_{g-1}^2 - 2u_g u_{g-2} = 0 \Rightarrow u_{g-1}^2 u_g - 2\underbrace{u_g^2 u_{g-2}}_{0} = 0$

$\Rightarrow u_{g-1}^2 u_g$.

Continue inductively.

Claim u_ε generate R_g as a $\mathbb{Q}$-vector space.

where $u_\varepsilon = u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_g^{\varepsilon_g}$, $\varepsilon_j \in \{0,1\}$

We induct on g. If $r \in R_g$, then

$$r = \text{elt in } R_{g-1} + u_g \cdot \text{elt } R_{g-1} + \cancel{u_g^2} \cdot \text{elt in } R_{g-1} + \cdots$$

0.

& apply induction

Claim u_ε give a basis of R_g as $\mathbb{Q}$-vector space.

Assume $\sum_\varepsilon a_\varepsilon u_\varepsilon = 0$. Order ε's lexicographically.

Thus $\varepsilon' \geq \varepsilon \iff$

$$\begin{array}{cccccccc} & g & g-1 & \cdots & k & \cdots & 1 \\ \varepsilon' = & * & * & \cdots & 1 & \cdots & \\ \varepsilon = & * & * & \cdots & 0 & \cdots & \end{array}$$

Define $\varepsilon^\perp = (1,1,\ldots 1) - \varepsilon$. If $\varepsilon' > \varepsilon$ then $u_{\varepsilon'} u_{\varepsilon^\perp} = 0$.

Indeed,

$$\begin{array}{cccccccc} & g & g-1 & \cdots & k & \cdots & 1 \\ \varepsilon' = & * & * & \cdots & 1 & \cdots & \\ \varepsilon = & * & * & \cdots & 0 & \cdots & \\ \varepsilon^\perp = & 1-* & 1-* & \cdots & 1 & \cdots & \end{array}$$

$\Rightarrow u_{\varepsilon'} u_{\varepsilon^\perp} = u_g \cdots u_{k+1} u_k^2 \cdots = 0$. while

$u_\varepsilon u_{\varepsilon^\perp} = u_1 \cdots u_g \neq 0$ (see below). Now if $\sum_\varepsilon a_\varepsilon u_\varepsilon = 0$

let ε be the smallest such that $a_\varepsilon \neq 0$. From

$$\sum_{\varepsilon'} a_{\varepsilon'} u_{\varepsilon'} = 0 \Rightarrow \sum_{\varepsilon' \geq \varepsilon} a_{\varepsilon'} \underbrace{u_{\varepsilon'} u_{\varepsilon^\perp}}_{0 \text{ if } \varepsilon \neq \varepsilon'} = 0 \Rightarrow a_\varepsilon = 0 \quad \text{false!}$$

Thus u_ε gives a basis for R_g.

Claim $u_1 \dots u_g \neq 0$ in R_g.

Indeed grade R_g via $\deg u_i = i$. Then $R_g = \bigoplus_d R_g^d$ and

since $u_1^{\varepsilon_1} \dots u_g^{\varepsilon_g}$ generators $\Rightarrow$ $d \leq 1 + 2 + \dots + g = \frac{g(g+1)}{2}$

$0 \leq \varepsilon_i \leq 1$

Furthermore for $d = \frac{g(g+1)}{2}$, R_g^d is spanned by $u_1 \dots u_g$. If

$u_1 \dots u_g = 0 \Rightarrow R_g^d = 0 \Rightarrow u_1^{\frac{g(g+1)}{2}} = 0$. But $R_g \longrightarrow R^*(A_{g+1,}$

$u_1 \longrightarrow \lambda_1$

$\Rightarrow \lambda_1^{\frac{g(g+1)}{2}} = 0$ contradicting Thm B.

Claim R_g satisfies Poincaré duality.

Indeed in the basis $\{u_\varepsilon\}$, the product is given

by an invertible triangular matrix. since.

$$u_{\varepsilon'} \cdot u_{\varepsilon^\perp} = 0 \quad \text{if } \varepsilon' > \varepsilon$$

$$\neq 0 \quad \text{if } \varepsilon' = \varepsilon$$

Claim $R_{g-1} \longrightarrow R^*(A_g)$ is an isomorphism.

$u_\varepsilon \longrightarrow \lambda_\varepsilon$

Define $\lambda_\varepsilon = \lambda_1^{\varepsilon_1} \cdots \lambda_{g-1}^{\varepsilon_{g-1}}$, $\varepsilon_i \in \{0,1\}$

λ_ε is also a basis for $R^*(A_g)$ by exactly the same argument as above. (Thm B holds for $R^*(A_g)$ and this is all we used).

Thus the above is an isomorphism.

Math 220B — Lecture 19

March 12, 2021

Last time

- $R_g = \mathbb{Q}[u_1, \dots u_g] \Big/ \text{Mumford relation}$

$$(1 + u_1 + u_2 + \dots + u_g)(1 - u_1 + u_2 - \dots \pm u_g) = 1.$$

R_g satisfies Poincaré duality

- $R^*(A_g) \cong R_{g-1}$

- only used Mumford relation & Theorem B:

$$\lambda_1^{\frac{g(g-1)}{2}} \neq 0 \quad \text{in} \quad R^*(A_g).$$

Remark Recall the Lagrangian Grassmannian $LG(g, \mathbb{C}^{2g})$.

$$LG = \{ \Lambda \subseteq \mathbb{C}^{2g}, \ \dim \Lambda = g, \ \Lambda \text{ Lagrangian} \}.$$

$\exists$ sequence over LG.

$$0 \longrightarrow S \longrightarrow \mathbb{C}^{2g} \otimes \mathcal{O} \longrightarrow Q \longrightarrow 0$$

$\Rightarrow$ using the symplectic form $Q \cong S^\vee$. Since

$$c(S)\, c(Q) = 1 \quad \Rightarrow \quad c(Q)\, c(Q^\vee) = 1.$$

Let $u_i = c_i(Q)$. The subring in $H^*(LG)$ generated by u_i satisfies $u_1^{\frac{g(g+1)}{2}} \neq 0$ so it is isomorphic to R_g.

In fact $\dim H^*(LG) = 2^g$ & $\dim R_g^\cdot = 2^g$ so

$$H^*(LG) = A^*(LG) \cong R_g.$$

Thm B $\lambda_1^{g(g-1)/2} \neq 0$ in $CH^*(\mathcal{A}_g)$.

Proof of thm B (sketch) WTS $\lambda_1^{\frac{g(g-2)}{2}} \neq 0$

(1) We show $\lambda_1^{\frac{g(g-1)}{2}} \neq 0$ over $\mathcal{A}_g \otimes \mathbb{F}_p$ $\forall p$ prime

(2) λ_1 is ample $/\mathbb{C}$ by Baily - Borel &

char p by Baily - Morel

(3) Find $Z \hookrightarrow \mathcal{A}_g \otimes \mathbb{F}_p$

- dimension $\frac{g(g-1)}{2}$ Koblitz
- complete Oort

Then $\lambda_1^{\frac{g(g-1)}{2}}\big/_Z \neq 0 \Rightarrow \lambda_1^{\frac{g(g-1)}{2}} \neq 0$.

What is Z ?

Recall A abelian variety $/\mathbb{C}$ then

$n : A \longrightarrow A$, $A[n] = \ker n \Rightarrow \# A[n] = n^{2g}$.

A abelian variety in char $= p$

This fails for $n = p$

$A/\mathbb{F}_p$ has p-rank 0 if $A \otimes \overline{\mathbb{F}_p}$ has no nontrivial p-torsion points.

$$Z = \{ A : p\text{-rank of } A = 0 \} \hookrightarrow \mathcal{A}_g \otimes \mathbb{F}_p .$$

Complete subvarieties in char 0.

Note $\lambda_1^{\frac{g(g-1)}{2}+1} = 0$ in $R_{g-1} \cong H^*(LG_{g-1})$

If Z complete subvariety of A_g of dim $= \frac{g(g-1)}{2}+1$.

$\lambda_1^{\frac{g(g-1)}{2}+1}/_Z = 0$. contradicting ampleness of λ_1.

$\Rightarrow \dim Z \leq \frac{g(g-1)}{2}$.

Oort's Conjecture

$\dim Z < \frac{g(g-1)}{2} \quad \forall\, Z \hookrightarrow A_g$ complete.

Solved by Sudan & Keel.

§ 1. K3 surfaces

$t = c_2(T^{\pi})$ $\qquad \pi : \mathcal{X} \longrightarrow \mathcal{F}_g.$ $\qquad \lambda = c_1(\mathbb{E})$

$\kappa_n = \pi_* \, t^{n+1}$

$$R^*(\mathcal{F}_g) = \mathbb{Q}[\lambda, \kappa] \big/ \text{Relations} \cong \mathbb{Q}[\lambda] \big/ \text{Relation}$$

Theorem (van der Geer, Katsura)

$$\kappa_n = a_n \lambda^{2n} \text{ where } \sum_{n=0}^{\infty} a_n x^n = 24 + 88x + 184x^2 + \dots$$

Proof GRR to $\pi : \mathcal{X} \longrightarrow \mathcal{F}_g$ and $\mathcal{O}_{\mathcal{X}}$.

$$\operatorname{ch} \pi_! \mathcal{O}_{\mathcal{X}} = \pi_*\left(\operatorname{ch} \mathcal{O}_{\mathcal{X}} \cdot \operatorname{Todd}(T^{\pi})\right) \qquad \text{GRR}$$

$$= \pi_*\left(\operatorname{Todd} T^{\pi}\right).$$

$\pi_! \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{F}} + \mathbb{E}^{\vee}$. Let r_1, r_2 be the roots of T^{π}.

$$\Rightarrow 1 + e^{-\lambda} = \pi_*\left(\frac{r_1}{1-e^{-r_1}} \cdot \frac{r_2}{1-e^{-r_2}}\right).$$

$$r_1 + r_2 = c_1(T^\pi) = -c_1(K_\pi) = -\pi^*\lambda$$

$$r_1 r_2 = c_2(T^\pi) = t$$

$$1 + e^{-\lambda} = \pi_*\left(\frac{r_1}{1-e^{-r_1}} \cdot \frac{r_2}{1-e^{-r_2}}\right).$$

Write $$\frac{r_1}{1-e^{-r_1}} \cdot \frac{r_2}{1-e^{-r_2}} = \sum_{j \le n/2} c_{n,j} \, (r_1+r_2)^{n-2j} (r_1 r_2)^j$$

$$= \sum_{j \le n/2} c_{n,j} \, (-1)^n \, \pi^* \lambda^{n-2j} \cdot t^j$$

$$\Rightarrow \pi_*\left(\frac{r_1}{1-e^{-r_1}} \cdot \frac{r_2}{1-e^{-r_2}}\right) = \sum_{j \le n/2} c_{n,j} \, (-1)^n \, \lambda^{n-2j} \cdot \kappa_{j-1}.$$

Coeff of λ^{2m-2} yields

$$\sum_{j \le m} c_{2m,j} \, \lambda^{2m-2j} \, \kappa_{j-1} = \frac{\lambda^{2m-2}}{(2m-2)!}$$

We establish the theorem by induction. If we know $\kappa_0, \dots, \kappa_{m-2}$ are of the form const $\times \lambda^s$ we solve for κ_{m-1} to conclude. We need $c_{2m,m} \neq 0$.

Claim $C_{2m,m} \neq 0$ Set $r_1 = r,\ r_2 = -r$

Proof

$$\frac{r}{1-e^{-r}} \cdot \frac{-r}{1-e^{r}} = \sum_m C_{2m,m}\, r^{2m} (-1)^m.$$

$$\left(1 + \frac{r}{2} + \sum_{k\geq 1} (-1)^{k-1} \frac{B_{2k}}{(2k)!} r^{2k}\right)\left(1 - \frac{r}{2} + \sum_{k\geq 1} (-1)^{k-1} \frac{B_{2k}}{(2k)!} r^{2k}\right)$$

$$\Longrightarrow C_{2m,m} = \sum_{i+j=m} \frac{B_{2i}}{(2i)!} \cdot \frac{B_{2j}}{(2j)!} \neq 0.$$

Question Can this method yield results over other moduli spaces?

Enriques ...

Bielliptics ...

Outcome $R^*(\mathcal{F}_g) = \mathbb{Q}[\lambda]/\text{Relation}.$

Theorem (van der Geer, Katsura).

$$R^* \mathcal{F}_g = \mathbb{Q}[\lambda]/\lambda^{18}=0. \Longrightarrow \text{Poincaré duality.}$$

Sketch of Proof

(1) $\lambda^{17} \neq 0$. Suffices to exhibit $Z \hookrightarrow \mathcal{F}_g$ complete of dim 17. This is because $\mathcal{F}_g \hookrightarrow \overline{\mathcal{F}_g}^{BB} \hookrightarrow \mathbb{P}^N$.

boundary is 1 dim'l. Define $Z = \overline{\mathcal{F}}^{L} \cap H_1 \cap H_2 \hookrightarrow \mathcal{F}$.

(2) $\lambda^{18} = 0$.

Let (L, v) be a lattice of type $(1, r-1)$ and

v primitive, $v^2 = 2g-2$.

$$\mathcal{F}_{(L,v)} = \{ (X, H, j) : j : L \hookrightarrow \mathrm{Pic}(X) \text{ primitive}, \ j(v) = \underset{\text{big \& nef.}}{H} \}.$$

$$\mathcal{F}_{(L,v)} \longrightarrow \mathcal{F}_g.$$

$$(X, H, j) \longrightarrow (X, H).$$

$$\dim \mathcal{F}_{L,v} = 20 - r.$$

(1) Borcherds relation:

$$\lambda \cdot \big/_{\mathcal{F}_{L,v}} = \sum_{L' \text{ type } (1,r)} c_{L,L',v,v'} \, \mathcal{F}_{L',v'}.$$

Induction on r: $\lambda^{19-r} = 0$ on $\mathcal{F}_{L,v}$

$$\lambda^{19-r} \big/_{\mathcal{F}_{L,v}} = \sum c \cdot \lambda^{18-r} \big/_{\mathcal{F}_{L',v'}} = 0.$$

---

(2) Base case $r = 17$ $\quad L \longrightarrow (1,16)$. Want $\lambda^2 = 0$.

$$\overline{\underline{IV}}_3 \cong \mathfrak{S}_2. \Rightarrow \mathcal{F}_{(L,v)} \longrightarrow \text{moduli of abelian surfaces}$$

(Siegel modular 3-fold)

Mumford relation: $(1 - \lambda_1 + \lambda_2)(1 + \lambda_1 + \lambda_2) = 0$

$$\Rightarrow \lambda_1^2 = 2\lambda_2 = 0.$$

§2. A richer tautological ring (2nd attempt)

$$K_{a,b} = \pi_*(\ell^a \cdot t^b), \quad \ell = c_1(\mathcal{L}), \quad t = c_2(T^\pi)$$

$$\begin{tikzcd} \mathcal{L} \arrow[r] & \mathcal{X} \arrow[d, "\pi"] \\ \mathcal{M} \arrow[r] & \mathcal{F} \end{tikzcd}$$

$K_{0,b}$ canonical.

Issue: $\mathcal{L} \longrightarrow \mathcal{L} \otimes \pi^* \mathcal{M}$ is not unique, $m = c_1(\mathcal{M})$.

$$\tilde{K}_{3,0} \longrightarrow K_{3,0} + (6g-6)\, m \qquad \pi_*\, c_1(\mathcal{L})^3$$

$$\tilde{K}_{1,1} \longrightarrow K_{1,1} + 24\, m \qquad \pi_*\, c_1(\mathcal{L})\, c_2(T^\pi).$$

The class $\gamma = K_{3,0} - \frac{g-1}{4} K_{1,1}$ is canonical.

Instead: $\ell = c_1(\mathcal{L})$. Work with

Better $\bar{\ell} = c_1(\mathcal{L}) - \frac{1}{g+1}\, \pi^* c_1(\pi_* \mathcal{L}) \in A^1(\mathcal{X})$.

$$\bar{K}_{3,0} = \frac{2}{g+1}\gamma - \frac{g+1}{2}\lambda \qquad \bar{K}_{a,b} = \pi_*(\bar{\ell}^a \cdot t^b).$$

$$\bar{K}_{1,1} = -\frac{4}{g+1}\gamma - \frac{2(g+1)}{g-1}\lambda.$$

Define $K^*(\mathcal{F}_g) = \mathbb{Q}[\bar{\kappa}_{a,b}] / \text{Relations}.$

- codimension 1 it contains λ and γ.
- codimension 2 ...

Question How do we find relations?

Remark

- Borcherds :

λ = is expressible in terms of codim 1. Noether - Lefschetz classes

- Farkas - Rimanyi (2018)

γ = is expressible in terms of codim 1. Noether - Lefschetz classes

Proof in degree 4 (Marian – O – 2012)

$\gamma = -10\lambda$. away from the loci P, Q, S

Idea $(X, \mathcal{L})$ is normally generated away from P, Q, S

$$\Longrightarrow \quad \mathrm{Sym}^2 H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^{\otimes 2}).$$

Note $\mathrm{rk}\, H^0(\mathcal{L}) = 2 + \frac{\mathcal{L}^2}{2} = 4$, $\mathrm{rk}\, \mathrm{Sym}^2 H^0(\mathcal{L}) = 10$

$\mathrm{rk}\, H^0(\mathcal{L}^2) = 2 + 2\mathcal{L}^2 = 10$

$$\Longrightarrow \quad \mathrm{Sym}^2 \pi_* \mathcal{L} \cong \pi_* \mathcal{L}^{\otimes 2} \text{ over } \mathcal{F} \setminus (P \cup Q \cup S)$$

Thus $c_1(\pi_* \mathcal{L}^2) = c_1(\mathrm{Sym}^2 \pi_* \mathcal{L}) = 5 c_1(\pi_* \mathcal{L})$. By

$$c_1(\pi_* \mathcal{L}) = -2\lambda + \frac{\kappa_{1,1}}{12} + \frac{\kappa_{3,0}}{6}$$

$$c_1(\pi_* \mathcal{L}^2) = -5\lambda + \frac{\kappa_{1,1}}{6} + \frac{4}{3}\kappa_{3,0}$$

$$\Longrightarrow \gamma = -10\lambda$$

In fact $-\gamma = \frac{22}{9} P + \frac{16}{27} Q + \frac{5}{27} S$ (see my website)

$\Longrightarrow$ γ, λ are supported on Noether Lefschetz loci

An even larger ring (3rd attempt)

- $NL^*(\mathcal{F}_g)$ = ring generated by $[\mathcal{F}_{(L,v)}]$.

- $R^*(\mathcal{F}_g)$ = ring generated by κ-classes

$\kappa_{a_1\dots a_r, b}$ from all NL - loci

Fix basis $v_1 \dots v_r$ of L. $\overset{j}{\rightsquigarrow}$ $\mathcal{H}_1 \dots \mathcal{H}_r \to \mathcal{X}_{L,v}$

$$\mathcal{X}_{L,v} \xrightarrow{\pi} \mathcal{F}_{L,v}.$$

Define

$$\kappa_{a_1\dots a_r\, b} = j_* \pi_* \left(c_1(\overline{\mathcal{H}_1})^{a_1} \dots c_1(\overline{\mathcal{H}_r})^{a_r} c_2(T_\pi)^b \right).$$

Clearly $NL^* \hookrightarrow R^*$.

<u>Conjecture</u> $NL^* = R^*$ (version of a conjecture in Marian-O-Pandharipande).

Remark $\exists$ different normalization $\bar{\mathcal{L}}$ (Pandharipande–Yin.)

$$\bar{\mathcal{L}} = \frac{1}{N} ev_* [\bar{M}_{0,1}(\mathcal{X}/\mathcal{F}, \mathcal{L})]^{red} \in A^1(\mathcal{X}).$$

$$N = \int_{[\bar{M}_{0,0}(X, L)]^{red}} 1$$

With the new normalization $\bar{R}^*(\mathcal{F}_g)$. It differs from $R^*(\mathcal{F}_g)$ only in codim 18 & 19 because the difference between $\bar{\mathcal{L}}$ and $\bar{\ell}$ is in general NL unless rk $L \geq 18$.

Why? $Pic(\mathcal{F}_L) = NL^1(\mathcal{F}_L)$ if rk $L \geq 18$

Question Is it true that $R^{18} = R^{19} = 0$?

Theorem (Petersen) True in cohomology.

$$RH^{18} = RH^{19} = 0.$$

Thus $R^*H = \overline{R}^*H$ and $R^* = \overline{R}^*$ conjecturally.

Theorem (Pandharipande - Yin).

$$\overline{R}^* \cong NL^*. \qquad R^* \cong NL^* \text{ in cohomology}$$

Conclusion Many Open Questions.