Problem 1.

Find the critical points of the function
\[ f(x, y) = x^3 + 6xy + 3y^2 - 9x \]
and determine their nature.

We calculate
\[ f_x = 3x^2 + 6y - 9, \quad f_y = 6x + 6y. \]
Setting \( f_y = 0 \) we find \( y = -x \).

Using \( f_x = 0 \) we find
\[ 3x^2 - 6x - 9 = 0 \implies x^2 - 2x - 3 = 0 \implies x = -1 \text{ or } x = 3. \]

We find the critical points \((-1, 1)\) and \((3, -3)\). To find the nature of the critical points we use the second derivative test. We have
\[ f_{xx} = 6x, \quad f_{yy} = 6, \quad f_{xy} = 6. \]

At the point \((-1, 1)\), we have \( H_f = \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix} \) which has negative determinant hence
\((-1, 1) \text{ is a saddle point} \).

At the point \((3, -3)\), we have \( H_f = \begin{bmatrix} 18 & 6 \\ 6 & 6 \end{bmatrix} \) which has negative determinant hence
\((3, -3) \text{ is a local minimum} \).
Problem 2.

Find the minimum and the maximum of the function

\[ f(x, y) = x^2 + 2y^2 - 6x + 2 \]

along the ellipse

\[ 2x^2 + y^2 = 8. \]

Write \( g(x, y) = 2x^2 + y^2 \). We have \( \nabla g = (4x, 2y) \neq (0, 0) \) when \( g = 8 \). Hence

\[ \nabla f = \lambda \nabla g \implies (2x - 6, 4y) = \lambda (4x, 2y). \]

The second equation gives

\[ 4y = 2\lambda y \]

hence either \( y = 0 \) or \( \lambda = 2 \). When \( \lambda = 2 \), we obtain

\[ 2x - 6 = 4\lambda x = 8x \implies x = -1 \implies y^2 = 6 \implies f(-1, \pm\sqrt{6}) = 21. \]

When \( y = 0 \) we obtain

\[ 2x^2 = 8 \implies x = \pm 2 \implies f(2, 0) = -6, f(-2, 0) = 18. \]

Thus \((-1, \pm\sqrt{6})\) gives the maximum value \([21]\) and \((2, 0)\) gives the minimum value \([-6]\).
Problem 3.

Assume that \( z = f(x, y) \) and
\[
x = \frac{u^2}{v}, \ y = 2uv - v^2.
\]

We are given
\[
f_x(4, 3) = 2, \ \frac{\partial f}{\partial y} = x - 2y.
\]
Calculate
\[
\frac{\partial z}{\partial v}
\]
at the point \( u = 2 \) and \( v = 1 \).

Note that when \( u = 2, v = 1 \) we must have \( x = 4 \) and \( y = 3 \). Hence
\[
\frac{\partial z}{\partial x} = 2 \text{ and } \frac{\partial z}{\partial y} = x - 2y \implies \frac{\partial z}{\partial y} = -2 \text{ at } x = 4, y = 3.
\]

Now
\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.
\]
We have
\[
\frac{\partial x}{\partial v} = -\frac{u^2}{v^2} \implies \frac{\partial x}{\partial v} = -4 \text{ at } u = 2, v = 1.
\]
Finally,
\[
\frac{\partial y}{\partial v} = 2u - 2v \implies \frac{\partial y}{\partial v} = 2 \text{ at } u = 2, v = 1.
\]
Therefore
\[
\frac{\partial z}{\partial v} = 2 \cdot (-4) + (-2) \cdot 2 = -12.
\]
Problem 4.

Find the volume of the region bounded on the top by the paraboloid \( z = 4 - x^2 - 3y^2 \), on the bottom by the \((x, y)\)-plane, on the sides by the planes \( x = 0 \), \( x = 1 \), \( y = -1 \) and \( y = 1 \).

We find
\[
\int_0^1 \int_{-1}^1 \left( 4 - x^2 - 3y^2 \right) \, dy \, dx.
\]
The inner integral equals
\[
\int_{-1}^1 \left( 4 - x^2 - 3y^2 \right) \, dy = \left[ 4y - x^2y - \frac{1}{3}y^3 \right]_{-1}^1 = 8 - 2x^2 - 2 = 6 - 2x^2.
\]
The outer integral equals
\[
\int_0^1 \left( 6 - 2x^2 \right) \, dx = 6x - \frac{2}{3}x^3 \bigg|_0^1 = 6 - \frac{2}{3} = \frac{16}{3}.
\]
Problem 5.

Consider the function
\[ f(x, y) = x^2 y^4 + xy^2 \ln(2x - y). \]

(i) Find the unit direction of steepest increase for the function \( f \) at the point \( P \).

We have
\[ f_x = 2xy^4 + y^2 \ln(2x - y) + xy^2 \cdot \frac{2}{2x - y} \quad \Rightarrow \quad f_x(1, 1) = 4. \]

Next,
\[ f_y = 4x^2 y^3 + 2xy \ln(2x - y) + xy^2 \cdot \frac{-1}{2x - y} \quad \Rightarrow \quad f_y(1, 1) = 3. \]

Thus \( \nabla f(1, 1) = (4, 3) \). The direction of steepest increase is \((4, 3)\).

(ii) Calculate the directional derivative \( D_\vec{v}f(P) \) in the direction \( \vec{v} = \frac{1}{\sqrt{2}} \vec{i} - \frac{1}{\sqrt{2}} \vec{j} \),

We have
\[ D_\vec{v}f(P) = \nabla f \cdot \vec{v} = (4, 3) \cdot (1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{\sqrt{2}}. \]

(iii) Calculate the tangent plane to the graph of \( f \) at the point \((P, f(P))\).

We have \( f(P) = 1 \) hence
\[ z - 1 = 4(x - 1) + 3(y - 1) \quad \Rightarrow \quad z = 4x + 3y - 6. \]

(iv) Find the tangent plane to the surface \( z^2 x^3 - f(x, y) = 0 \) at the point \((1, 1, 1)\).

We set \( g(x, y, z) = z^2 x^3 - f(x, y) \). We find
\[ g_x = 3x^2 z^2 - f_x \quad \Rightarrow \quad g_x(1, 1, 1) = 3 - 4 = -1 \]
\[ g_y = -f_y \quad \Rightarrow \quad g_y(1, 1, 1) = -3 \]
\[ g_z = 2zx^3 \quad \Rightarrow \quad g_z(1, 1, 1) = 2. \]

The tangent plane is
\[ -(x - 1) - 3(y - 1) + 2(z - 1) = 0 \quad \Rightarrow \quad -x - 3y + 2z = -2. \]