Problem 1.

Evaluate by changing the order of integration

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy.$$

`Solution:` *We change the order of integration over the region*

$$0 \leq \sqrt{y} \leq x \leq 1.$$

We find

$$0 \leq y \leq x^2,\ 0 \leq x \leq 1.$$

and

$$\int_0^1 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^1 x^2 e^{x^3}\, dx = \frac{1}{3} e^{x^3} |_{x=0}^{x=1} = \frac{1}{3}(e-1).$$

Problem 2.

Find the critical points of the function

$$f(x,y) = xy^2 - 4xy + \frac{1}{2}x^2$$

and determine their nature.

`Solution:` *We calculate*

$$f_x = y^2 - 4y + x, \; f_y = 2xy - 4x = 2x(y-2).$$

Since

$$f_y = 0 \implies x = 0 \text{ or } y = 2.$$

When $x = 0$ *we obtain from*

$$f_x = 0 \implies y^2 - 4y = 0 \implies y = 0 \text{ or } y = 4.$$

When $y = 2$ *we find*

$$f_x = 0 \implies x = 4.$$

There are three critical points $(0,0)$, $(0,4)$ *and* $(4,2)$.

We find the second derivatives

$$f_{xx} = 1, \; f_{xy} = 2y - 4, \; f_{yy} = 2x.$$

At the points where $x = 0$, *the determinant of the hessian is*

$$D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 1 \cdot 0 - (2y-4)^2 < 0$$

hence both $(0,0)$ *and* $(0,4)$ *are saddle points. For* $(4,2)$, *the determinant of the Hessian is*

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 1 \cdot 8 - 0^2 > 0$$

hence $(4,2)$ *is a local minimum.*

Problem 3.

Consider the function

$$f(x,y) = \frac{2x^2y}{x^4+y^2}.$$

(i) Show that the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

along any line of fixed slope m through the origin equals 0.

(ii) Evaluate the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ along the parabola $y = x^2$.

(iii) What is the value of the limit $\lim_{(x,y)\to(0,0)} f(x,y)$?

`Solution:`

(i) *We have* $y = mx$ *and*

$$f(x,y) = \frac{2x^2(mx)}{x^4+(mx)^2} = \frac{2mx}{x^2+m^2} \to \frac{0}{m^2}$$

provided $x \to 0$. *Thus* $f(x,y) \to 0$ *along the line* $y = mx$ *provided* $m \neq 0$. *When* $m = 0$, *we have* $y = 0$ *and* $f(x,y) = 0$.

(ii) *When* $y = x^2$, *we have* $f(x,y) = \frac{2x^2y}{x^4+y^2} = 1$, *so the limit equals* 1.

(iii) *Since the answers in (i) and (ii) are different, the limit does not exist.*

Problem 4.

Consider the function

$$f(x,y) = \sqrt{6 - x^2 - y^2}.$$

(i) Find the direction of steepest increase of f at the point $P(1,2)$.
(ii) Draw the graph of the function f.
(iii) Find the directional derivative $D_{\vec{v}}f(1,2)$ in the direction $\vec{v} = \frac{3}{5}\vec{\imath} - \frac{4}{5}\vec{\jmath}$.
(iv) Find the linear approximation $f((1,2) + .01 \cdot \vec{v})$.
(v) Find the tangent plane to the surface

$$z^2 x + z f(x^2, y) = 2$$

at $(1,2,1)$.

`Solution:`

(i) *We have*

$$f_x = \frac{1}{2} \cdot \frac{-2x}{\sqrt{6 - x^2 - y^2}} = -\frac{x}{\sqrt{6 - x^2 - y^2}} \implies f_x(1,2) = -1.$$

Similarly,

$$f_y = -\frac{y}{\sqrt{6 - x^2 - y^2}} \implies f_y(1,2) = -2.$$

Thus $\nabla f = (-1,-2)$ *and the direction of steepest increase is* $(-1,-2)$.

(ii) *The graph is*

$$z = f(x,y) = \sqrt{6 - x^2 - y^2} \implies x^2 + y^2 + z^2 = 6,\ z \geq 0$$

which is the half sphere of radius $\sqrt{6}$.

(iii) *We have*

$$D_{\vec{v}}f = \nabla f \cdot v = (-1,-2) \cdot (3/5, -4/5) = 1.$$

(iv) *We evaluate*

$$f((1,2) + .01\vec{v}) \approx f(1,2) + .01 \cdot D_{\vec{v}}f = 1 + .01 \cdot 1 = 1.01.$$

(v) *We have*

$$g(x,y,z) = z^2 x + z f(x^2, y)$$

hence

$$g_x = z^2 + 2zx f_x(x^2, y) \implies g_x(1,2,1) = 1 + 2(-1) = -1$$
$$g_y = z f_y(x^2, y) \implies g_y(1,2,1) = 1(-2) = -2$$
$$g_z = 2zx + f(x^2, y) \implies g_z(1,2,1) = 3.$$

The normal vector is $(-1,-2,3)$ *hence the plane is*

$$-x - 2y + 3z = -2.$$

Problem 5.

Find the minimum and the maxim value of the function

$$f(x, y, z) = 2x - 2y + z$$

along the sphere of center $(1, 0, -1)$ and radius 3.

`Solution:` *We have*

$$g(x, y, z) = (x-1)^2 + y^2 + (z+1)^2 = 9.$$

We need

$$\nabla f = \lambda \nabla g$$

which gives

$$(2, -2, 1) = 2\lambda(x-1, y, z+1) \implies x - 1 = \frac{2}{2\lambda}, y = \frac{-2}{2\lambda}, z + 1 = \frac{1}{2\lambda}.$$

Using

$$(x-1)^2 + y^2 + (z+1)^2 = 9 \implies \left(\frac{2}{2\lambda}\right)^2 + \left(-\frac{2}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 9 \implies 2\lambda = \pm 1.$$

When $2\lambda = 1$ we find the point

$$(3, -2, 0) \text{ with } f(3, -2, 0) = 10.$$

When $2\lambda = -1$ we find the point

$$(-1, 2, -2) \text{ with } f(-1, 2, -2) = -8.$$

Problem 6.

Find the area of the region bounded in the first quadrant by the hyperbola $xy = 1$ and the parabolas $x = y^2$ and $x = 8y^2$. Express the answer in the simplest possible form.

`Solution:` *The intersection of the parabola* $x = y^2$ *and the hyperbola* $xy = 1$ *occurs at* $(1, 1)$. *The intersection of the parabola* $x = 8y^2$ *and the hyperbola* $xy = 1$ *is the point* $(2, \frac{1}{2})$. *The integral is to be split into two regions. For the region* $0 \le x \le 1$, *we have*

$$y_{min} = \sqrt{\frac{x}{8}},\ y_{max} = \sqrt{x}.$$

For the region $1 \le x \le 2$, *we have*

$$y_{min} = \sqrt{\frac{x}{8}},\ y_{max} = \frac{1}{x}.$$

We calculate

$$\begin{aligned} Area &= \int_0^1 \int_{\sqrt{x/8}}^{\sqrt{x}} dy\, dx + \int_1^2 \int_{\sqrt{x/8}}^{1/x} dy\, dx \\ &= \int_0^1 \sqrt{x} - \sqrt{\frac{x}{8}}\, dx + \int_1^2 \frac{1}{x} - \sqrt{\frac{x}{8}}\, dx \\ &= \frac{2}{3} x^{3/2} (1 - \frac{1}{\sqrt{8}})|_{x=0}^{x=1} + \ln x|_{x=1}^{x=2} - \frac{2}{3} \frac{x^{3/2}}{\sqrt{8}}|_{x=1}^{x=2} = \ln 2. \end{aligned}$$

Problem 7.

Find the average value of the function

$$f(x, y, z) = xyz$$

over the first octant

$$x \geq 0, y \geq 0, z \geq 0$$

of the ball $x^2 + y^2 + z^2 \leq 1$.

`Solution:` *We use spherical coordinates. We have*

$$\bar{f} = \frac{1}{volume} \int \int \int xyz \, dV.$$

Thus

$$\begin{aligned}
\bar{f} = \frac{1}{4\pi/3 \cdot 1/8} \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta \\
= \frac{6}{\pi} \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^5 \sin^3\phi \cos\phi \cos\theta \sin\theta \, d\rho \, d\phi \, d\theta \\
= \frac{6}{\pi} \int_0^1 \rho^5 \, d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^3\phi \cos\phi \, d\phi \cdot \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta \\
= \frac{6}{\pi} \cdot \frac{\rho^6}{6} \Big|_{\rho=0}^{\rho=1} \cdot \frac{1}{4} \sin^4\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \cdot \frac{1}{2} \sin^2\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{6}{\pi} \cdot \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8\pi}.
\end{aligned}$$

Problem 8.

Using cylindrical coordinates, find the mass of the solid with density $\rho = z$ bounded by the sphere $x^2 + y^2 + z^2 = 2$ and the cone $z^2 = x^2 + y^2$.

`Solution:` *We calculate*

$$\int\int\int \rho \, dV.$$

The intersection is

$$x^2 + y^2 = z^2 = 1$$

and therefore the cylindrical coordinates satisfy

$$0 \le r \le 1, r \le z \le \sqrt{2 - r^2}.$$

We have

$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} z \, dz (r \, dr) \, d\theta = 2\pi \int_0^1 \frac{1}{2} z^2 |_{z=r}^{z=\sqrt{2-r^2}} r \, dr =$$
$$= 2\pi \int_0^1 (1 - r^2) r \, dr = 2\pi (\frac{r^2}{2} - \frac{r^4}{4})|_{r=0}^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

Problem 9.

Find the volume of the solid bounded by the cylinders

$$z = 1 - y^2, \ z = y^2 - 1$$

and the planes $x = 0$ and $x + z = 1$.

`Solution:` *We calculate*

$$\int_{-1}^{1} \int_{y^2-1}^{1-y^2} \int_{0}^{1-z} dx\, dz\, dy = \int_{-1}^{1} \int_{y^2-1}^{1-y^2} (1-z)\, dz\, dy = \int_{-1}^{1} (z - \frac{1}{2}z^2)|_{z=y^2-1}^{z=1-y^2}\, dy$$

$$= \int_{-1}^{1} 2(1-y^2)\, dy = 2y - \frac{2}{3}y^3|_{y=-1}^{y=1} = 4 - \frac{4}{3} = \frac{8}{3}.$$

Problem 10.

Consider the parametric curve given by

$$\vec{r}(t) = \left(\frac{t^4}{4}, \frac{t^6}{6}\right), \quad 0 \le t \le 1.$$

Find the arclength parametrization of the curve.

`Solution:` *We have*

$$\vec{r}'(t) = (t^3, t^5) \implies ||\vec{r}'(t)|| = \sqrt{(t^3)^2 + (t^5)^2} = t^3\sqrt{1+t^4}.$$

The arclength function is

$$s(t) = \int_0^t ||\vec{r}'(u)||\, du = \int_0^t u^3\sqrt{1+u^4}\, du = \frac{1}{6}(1+u^4)^{\frac{3}{2}}|_{u=0}^{u=t} = \frac{(1+t^4)^{\frac{3}{2}} - 1}{6}.$$

Solving $s(t) = s$ we find

$$\frac{(1+t^4)^{\frac{3}{2}} - 1}{6} = s \implies t^4 = (6s+1)^{\frac{2}{3}} - 1.$$

Thus

$$\vec{r}(t) = \left(\frac{1}{4}((6s+1)^{\frac{2}{3}} - 1), \frac{1}{6}((6s+1)^{\frac{2}{3}} - 1)^{\frac{3}{2}}\right).$$

Finally, $t = 0$ corresponds to $s = 0$ and $t = 1$ corresponds to $s = \frac{1}{6}(2^{\frac{3}{2}} - 1)$.

Problem 11.

Assume that
$$w = \ln(x^2 - y^2 + z^2)$$
where
$$x = 2s + t,\ y = 2s - t,\ \ z = 2\sqrt{st}.$$
Using the chain rule, calculate the derivative
$$\frac{\partial w}{\partial s}.$$
Express your answer in the simplest possible form.

`Solution:` *We have*
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}.$$
We compute
$$\frac{\partial w}{\partial x} = \frac{2x}{x^2 - y^2 + z^2} = \frac{2(2s+t)}{(2s+t)^2 - (2s-t)^2 + 4st} = \frac{2(2s+t)}{12st} = \frac{2s+t}{6st}.$$
Similarly
$$\frac{\partial w}{\partial y} = -\frac{(2s-t)}{6st}$$
$$\frac{\partial w}{\partial z} = \frac{4\sqrt{st}}{12st} = \frac{1}{3\sqrt{st}}.$$
Next,
$$\frac{\partial x}{\partial s} = 1,\ \frac{\partial y}{\partial s} = 1,\ \frac{\partial z}{\partial s} = 2 \cdot \frac{1}{2\sqrt{s}} \cdot \sqrt{t} = \sqrt{\frac{t}{s}}.$$
Therefore
$$\frac{\partial w}{\partial s} = \frac{(2s+t)}{6st} \cdot 2 + \frac{-(2s-t)}{6st} \cdot 2 + \frac{1}{3\sqrt{st}} \cdot \sqrt{\frac{t}{s}} = \frac{4t}{6st} + \frac{1}{3s} = \frac{1}{s}.$$

Problem 12.

The ellipsoid $x^2 + 2y^2 + z^2 = 4$ and the plane $2x + y + 3z = 6$ intersect in an ellipse passing through the point $(1, 1, 1)$. Find the parametrization of the tangent line to the ellipse at $(1, 1, 1)$.

`Solution:` *The normal vector to the ellipsoid is* $\vec{n}_1 = (2x, 4y, 2z)$ *which gives at* $(1, 1, 1)$ *the vector*

$$\vec{n}_1 = (2, 4, 2).$$

The normal vector to the plane is

$$\vec{n}_2 = (2, 1, 3).$$

The vector tangent to the ellipse is perpendicular to both $\vec{n}_1$ *and* $\vec{n}_2$ *hence*

$$\vec{t} = \vec{n}_1 \times \vec{n}_2 = (2, 4, 2) \times (2, 1, 3) = (10, -2, -6).$$

The tangent line is

$$(1, 1, 1) + t(10, -2, -6).$$