Problem 1.

Consider the points $P(1,1,-2)$, $Q(2,0,1)$ and $R(1,-1,0)$.

(i) Find the area of the triangle PQR.

We calculate

$$\vec{PQ} = (1,-1,3),\ \vec{RQ} = (1,1,1).$$

The area of the parallelogram spanned by $\vec{P}Q$ and $\vec{Q}R$ is the magnitude of the cross product

$$(2,-1,3) \times (1,1,1) = (-4,2,2).$$

This vector has magnitude $2\sqrt{6}$, so the triangle PQR has area $\sqrt{6}$.

(ii) Find the equation of the plane through P, Q and R.

The plane through P, Q and R has as normal vector the cross product. The entries of the cross product are used as coefficients for the plane. We obtain the equation

$$-4x + 2y + 2z = -6 \iff -2x + y + z = -3$$

using the point P (or Q or R) to find the right hand side.

Problem 2.

(i) Does there exist a constant such that the function

$$f(x,y,z) = \begin{cases} \frac{x^4 y}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ a & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

is continuous?

We must have

$$a = \lim_{(x,y,z)\to(0,0,0)} \frac{x^4 y}{x^2+y^2+z^2}.$$

Note that

$$0 \leq \left| \frac{x^4 y}{x^2+y^2+z^2} \right| = \frac{x^2}{x^2+y^2+z^2} \cdot |x^2 y| \leq |x^2 y| \to 0$$

hence $a = 0$.

(ii) Does the limit

$$\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^4+(y-1)^4}$$

exist?

We find the limit along the line $y-1 = mx$, keeping m fixed. Then, we obtain

$$\lim_{\substack{y-1=mx \\ x\to 0}} \frac{x^2(y-1)^2}{x^4+(y-1)^4} = \lim_{x\to 0} \frac{x^2 \cdot (mx)^2}{x^4+(mx)^4} = \frac{m^2}{m^4+1}.$$

Since this depends on m, the original limit does not exist.

Problem 3.

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors whose magnitudes are 1, 2, and 1 respectively. Suppose that $\vec{x}$ is parallel to (and in the same direction as) $\vec{y}$, and $\vec{x}$ is perpendicular to $\vec{z}$. Find the angle between the vectors $\vec{x}+\vec{z}$ and $\vec{y}+3\vec{z}$.

We have

$$\vec{x}\cdot\vec{x}=1,\ \vec{y}\cdot\vec{y}=4,\ \vec{z}\cdot\vec{z}=1.$$

Furthermore

$$\vec{x}\cdot\vec{y}=1\cdot 2\cos 0=2,\ \vec{x}\cdot\vec{z}=0,\ \vec{y}\cdot\vec{z}=0.$$

The last two products are justified since the vectors involved are perpendicular.

We compute

$$(\vec{x}+\vec{z})\cdot(\vec{y}+3\vec{z})=\vec{x}\cdot\vec{y}+\vec{z}\cdot\vec{y}+3\vec{x}\cdot\vec{z}+3\vec{z}\cdot\vec{z}=2+0+0+3=5.$$

We have

$$||\vec{x}+\vec{z}||^2=(\vec{x}+\vec{z})\cdot(\vec{x}+\vec{z})=\vec{x}\cdot\vec{x}+2\vec{x}\cdot\vec{z}+\vec{z}\cdot\vec{z}=1+0+1=2 \implies ||\vec{x}+\vec{z}||=\sqrt{2}$$

$$||\vec{y}+3\vec{z}||^2=(\vec{y}+3\vec{z})\cdot(\vec{y}+3\vec{z})=\vec{y}\cdot\vec{y}+6\vec{y}\cdot\vec{z}+9\vec{z}\cdot\vec{z}=4+0+9=13 \implies ||\vec{y}+3\vec{z}||=\sqrt{13}.$$

Thus

$$\cos\theta=\frac{6}{\sqrt{2}\cdot\sqrt{13}}=\frac{5}{\sqrt{26}} \implies \theta=\cos^{-1}\frac{5}{\sqrt{26}}.$$

Problem 4.

A line ℓ is perpendicular to the plane $x + 2y - 3z = 2$ and passes through the point $(1, 0, -1)$. Where does the line intersect the plane $x - 2y + z = -4$?

The line is parametrized by

$$(x, y, z) = (1, 0, -1) + t(1, 2, -3).$$

Then

$$x = 1 + t,\ y = 2t\ \ z = -1 - 3t$$

and since

$$x - 2y + z = 1$$

we must have

$$(1 + t) - 4t + (-1 - 3t) = -4 \implies -6t = -4 \implies t = 2/3.$$

This gives $(x, y, z) = (1, 0, -1) + \frac{2}{3}(1, 2, -3) = (\frac{5}{3}, \frac{4}{3}, -3)$.

Problem 5.

Consider the function $f(x,y) = 1 - \frac{1}{9}(x-1)^2 - \frac{1}{4}y^2$.

(i) Sketch the level diagram for f showing at least three level curves.

The level curves are

$$f(x,y) = c \iff 1 - \frac{1}{9}(x-1)^2 - \frac{1}{4}y^2 = c \iff \frac{1}{9}(x-1)^2 + \frac{1}{4}y^2 = 1 - c.$$

The level curves are ellipses centered at $(1,0)$. We can pick the levels $c = 0$, $c = -3$ and $c = -8$, for instance. We obtain the three ellipses

$$\frac{1}{9}(x-1)^2 + \frac{1}{4}y^2 = 1, \quad \frac{1}{9}(x-1)^2 + \frac{1}{4}y^2 = 4 \text{ and } \frac{1}{9}(x-1)^2 + \frac{1}{4}y^2 = 9.$$

(ii) Sketch the graph of f.

The graph of f will be a paraboloid whose z-cross sections are ellipses. The highest point on the paraboloid is $(1,0,1)$. The paraboloid is "concave down".

Problem 6.

The trajectory of a particle is given by the parametric curve

$$x = \cos t + 2\sin t, \quad y = 2\cos t - \sin t, \quad z = 2t, \quad 0 \le t \le \pi.$$

(i) Find the speed and velocity of the particle.

The velocity equals

$$\vec{r}'(t) = (-\sin t + 2\cos t, -2\sin t - \cos t, 2),$$

while the speed is

$$||\vec{r}'(t)|| = \sqrt{(-\sin t + 2\cos t)^2 + (-2\sin t - \cos t)^2 + 4}$$

$$= \sqrt{(\sin^2 t + 4\cos^2 t - 4\cos t \sin t) + (4\sin^2 t + \cos^2 t + 4\sin t \cos t) + 4}$$

$$= \sqrt{5(\sin^2 t + \cos^2 t) + 4} = \sqrt{9} = 3.$$

(ii) Calculate the tangent line to the trajectory at $t = \frac{\pi}{2}$.

We calculate

$$\vec{r}'(\frac{\pi}{2}) = (-1, -2, 2).$$

Also $r(\pi/2) = (2, -1, \pi)$. Thus the tangent line passes through $(2, -1, \pi)$ and is parallel to $(-1, -2, 2)$. The parametric equation is

$$(x, y, z) = (2, -1, \pi) + s(-1, -2, 2).$$

(iii) Calculate the arclength parametrization of the trajectory.

We solve for the arclength function

$$s(t) = \int_0^t 3\, du = 3t$$

and $s(t) = s \implies t = s/3$. This gives the parametrization

$$\vec{R}(s) = \vec{r}(s/3) = (\cos\frac{s}{3} + 2\sin\frac{s}{3}, 2\cos\frac{s}{3} - \sin\frac{s}{3}, \frac{2s}{3}).$$

(iv) Show that the trajectory stays on a cylinder whose central axis is the z-axis. Draw the trajectory of the particle.

We calculate

$$x^2 + y^2 = (\cos t + 2\sin t)^2 + (2\cos t - \sin t)^2 = 5(\sin^2 t + \cos^2 t) = 5.$$

This is the equation of a cylinder of radius $\sqrt{5}$ with central axis the z-axis. The trajectory is a helix wrapping around the cylinder.